Hilbert Spaces

An inner product space V carries a natural norm induced by its inner product. We say that V is a *Hilbert space* if V is complete with respect to this norm.

As seen earlier, $\mathbb{R}^n$ carries a natural inner product and is complete with respect to the induced norm. So it is a Hilbert space. More generally, we saw that any finite dimensional inner product space can be identified with $\mathbb{R}^n$ and its natural inner product. Thus, such a space is automatically complete. The primary object of our study is therefore infinite dimensional spaces.

As shown earlier, any normed space V can be embedded in its completion $\overline{V}$. A continuous linear functional on V extends to $\overline{V}$ in a unique way as a continuous linear functional there. Moreover, a continuous linear functional on $\overline{V}$ is uniquely determined by its restriction to the *dense* subspace V. In other words, the restriction map $B(\overline{V}, \mathbb{R}) \to B(V, \mathbb{R})$ is an isomorphism.

Exercise: For any continuous linear functional f on $\overline{V}$, show that

$$\|f\| = \{|f(v)| : v \in V \text{ and } |v| \leq 1\}$$

(Hint: Show that if v_n is a Cauchy sequence in V that converges to w in $\overline{V}$, then there is a Cauchy sequence w_n in V converging to w such that $\|w_n\| \leq \|w\|$.)

As shown earlier, the normed space $B(V, W)$ is complete with the operator norm whenever W is complete. Thus, $B(V, \mathbb{R})$ is a complete normed space.

For an inner product space V we have shown the Cauchy-Schwarz inequality:

$$|\langle v, w\rangle| \leq \|v\|\|w\|$$

Exercise: Show that the function $f_w : V \to \mathbb{R}$ defined by $f_w(v) = \langle v, w\rangle$ is continuous with respect to the norm induced by the inner product. Moreover, the "operator norm" $\|f_w\|$ satisfies $\|f_w\| = \|w\|$.

We thus get a natural map $f : V \to B(V, \mathbb{R})$ given by $v \mapsto f_v$ which preserves the norm.

Since $\overline{V}$ is the completion of V, we get a natural extension of this map to $g : \overline{V} \to B(V, \mathbb{R})$ which also preserves the norm. It follows that g is 1-1. Since $B(V, \mathbb{R}) = B(\overline{V}, \mathbb{R})$ it follows that for every pair of vectors a and b in $\overline{V}$, we can define $[a, b] = g(b)(a)$.

We see easily that the pairing $[,]$ is bi-linear. Moreover, if v_n is a Cauchy sequence in V that converges to a and w_n is a Cauchy sequence in V that converges to b, then

$$g(b)(a) = \lim_{n\to\infty} g(b)(v_n) = \lim_{n\to\infty} g(w_n)(a)$$

The first equality follows from the continuity of $g(b)$ and then the second equality follows from the continuity of g. Now, $g(w_n) = f(w_n)$, since g restricts to f on

V. By the continuity of $f(w_n)$ we have

$$f(w_n)(a) = \lim_{k\to\infty} f(w_n)(v_k)$$

By definition $f(w_n)(v_k) = \langle v_k, w_n\rangle$. It follows that

$$f(w_n)(v_k) = \langle v_k, w_n\rangle = \rangle w_n, v_k\rangle = f(v_k)(w_n) = g(v_k)(w_n)$$

Taking the limit as k goes to infinity we have

$$f(w_n)(a) = \lim_{k\to\infty} f(w_n)(v_k) = \lim_{k\to\infty} g(v_k)(w_n) = g(a)(w_n)$$

Now, take the limit as n goes to infinity to obtain

$$g(b)(a) = \lim_{n\to\infty} g(w_n)(a) = \lim_{n\to\infty} f(w_n)(a) = \lim_{n\to\infty} g(a)(w_n) = g(a)(b)$$

Thus, we see that $[,]$ on $\overline{V}$ is symmetric.

Exercise: Show that if $[v,v] = 0$ for v in $\overline{V}$, then $v = 0$. (Hint: If v_n is a sequence in V that converges to v, then show that $[v,v] = \lim n \to \infty\langle v_n, v_n\rangle$.)

Thus $[,]$ is an inner product on $\overline{V}$. In other words:

The completion of an inner product space is a complete inner product space, i.e. a Hilbert space.

The space ℓ_2.

One way to "construct" an infinite dimensional space is as follows. Consider the space $\mathbb{R}^\infty$ of "eventually 0" sequences; a sequence $(a_1, a_2, \dots)$ lies in $\mathbb{R}^\infty$ if there is an n so that $a_k = 0$ for $k > n$. If we consider $\mathbb{R}^n$ as the subset of $\mathbb{R}^\infty$ consisting all a so that $a_k = 0$ for $k > n$, then $\mathbb{R}^\infty$ is the union of $\mathbb{R}^n$ over all n.

Exercise: Check that $\mathbb{R}^\infty$ inherits the structure of a vector space over $\mathbb{R}$ from the vector space structure on $\mathbb{R}^n$.

Given two elements a and b in $\mathbb{R}^\infty$ we can define

$$\langle a, b\rangle = \sum_{k=1}^{\infty} a_k b_k$$

This sum "looks" infinite, but is actually *finite* as there is a common n so that a and b lie in $\mathbb{R}^n$.

Exercise: Check that this makes $\mathbb{R}^\infty$ into an inner product space. (Hint: Note that the identities that we need to check involve at most two vectors and both of those lie in $\mathbb{R}^n$ for some common n.)

This induces a norm on $\mathbb{R}^\infty$ which we denote as $\|\cdot\|_2$. The completion of $\mathbb{R}^\infty$ with respect to this norm is what we now want to study.

Let H be the space $B((\mathbb{R}^\infty, \|\cdot\|_2), \mathbb{R})$ of linear functionals on $\mathbb{R}^\infty$ that are bounded with respect to the norm $\|\cdot\|_2$. Given a linear functional $h \in H$ it is determined by its sequence of values $h(e^{(n)})$, where $e^{(n)}$ is the standard basis of $\mathbb{R}^\infty$. Hence, we can identify H as a subset (in fact, subspace) of the space of sequences. Let $b = (b_n)$ be a sequence in H and h_b denote the linear functional associated with it.

Exercise: For a vector $a = (a_n)$ in $\mathbb{R}^\infty$ show that $h_b(a) = \sum_{n=1}^\infty a_b b_n$. (Note that this sum is a finite sum since $a_n = 0$ for sufficiently large n.)

Since we are given that h_b is a bounded linear functional, there is a positive constant C so that $|h_b(a)| \leq C\|a\|_2$ for all a in $\mathbb{R}^\infty$. In particular, let is define for each N, a vector $a^{(N)}$ as follows:

$$a_n^{(N)} = \begin{cases} \frac{|b_n|^2}{b_n} & \text{if } b_n \neq 0 \text{ and } n \leq N \\ 0 & \text{if } b_n = 0 \text{ or } n > N \end{cases}$$

Exercise: Check that

$$h_b(a^{(N)}) = \sum_{n=1}^{N} |b_n|^2 = \langle a^{(N)}, a^{(N)} \rangle$$

Now, if b is *not* the sequence consisting of zeroes, then $a^{(N)}$ is a non-zero vector in $\mathbb{R}^\infty$ for sufficiently large N. Hence, we can define $v^{(N)} = a^{(N)}/\|a^{(N)}\|_2$ as the unit vector along $a^{(N)}$.

Exercise: Check that

$$h_b(v^{(N)}) = (\sum_{n=1}^{N} |b_n|^2)^{1/2} = \|a^{(N)}, a^{(N)}\|_2$$

Since we have $\|v^{(N)}\|_2 = 1$, the boundedness of h_b gives us $h_b(v^{(N)}) \leq C$. Thus, we see that $\left(\sum_{n=1}^{N} |b_n|^2\right) \leq C^2$ is bounded independent of N. In other words, we have show that $\sum_{n=1}^\infty |b_n|^2$ is a convergent sum.

The space ℓ_2 consists of sequences $(a_1, \ldots,)$ such that $\sum_{k=1}^\infty |a_k|^2$ is finite. We have now shown that H is contained in ℓ_2.

Exercise: Given *any* sequence $b = (b_n)$, we can define the linear functional $h_b : \mathbb{R}^\infty \to \mathbb{R}$ by the formula $h_b(a) = \sum_{n=1}^\infty a_b b_n$. (Note that this sum is a finite sum since $a_n = 0$ for sufficiently large n.)

However, this linear functional need not be continuous with respect to a given norm! Now, assume that b is in ℓ_2 and let $C = \sum_{n=1}^\infty \|b_N|^2$.

Exercise: For any a in $\mathbb{R}^n$, show that $|h_b(a)|^2 \leq C\|a\|_2^2$. (Hint: Use the Cauchy-Shwarz inequality for $\mathbb{R}^n$!)

Since $\mathbb{R}^\infty$ is the *union* of the subspaces $\mathbb{R}^n$, we see that h_b is a bounded linear function when b is in ℓ_2. In other words, we have shown that ℓ_2 is contained in H. Thus $H = \ell_2$.

Exercise: By following the above exercises carefully, show that the operator norm $\|h_b\|$ of h_b is given by $\left(\sum_{n=1}^\infty |b_n|^2\right)^{1/2}$.

It is therefore natural to define the norm on ℓ_2 by:

$$\|b\|_2 = \left(\sum_{n=1}^\infty |b_n|^2\right)^{1/2}$$

Above, we have shown that the completion of an inner-product space is a contained in its dual space. So, we see that the completion of $\mathbb{R}^\infty$ is contained in ℓ_2 in a natural way. We will now see that this is an equality. Given any sequence b in ℓ_2, we define $b^{(n)}$ to be the truncated sequence which consists of 0 beyond the n-th co-ordinate.

Exercise: Show that $\|b - b^{(N)}\|_2^2 = \sum_{n>N} |b_n|^2$.

Exercise: Given $\epsilon > 0$ show that there is an N so that if $n \leq N$, then $\|b - b^{(N)}\| < \epsilon$.

In other words, $b^{(n)}$ is a Cauchy sequence converging to b in ℓ_2. Moreover, $b^{(n)}$ lies in $\mathbb{R}^\infty$. We thus see that $\mathbb{R}^\infty$ is dense in ℓ_2. Since we have already seen that completion of $\mathbb{R}^\infty$ with respect to $\|\cdot\|_2$ is *contained* in ℓ_2, we obtain equality.

Above, we have shown that the completion of an inner-product space is also an inner-product space. This is one way to prove that ℓ_2 is an inner-product space. However, we can also do this more directly as follows.

Exercise: Given sequences a and b in ℓ_2, show that

$$|\sum_{k=N}^M a_k b_k|^2 \leq \left(\sum_{k=N}^M |a_k|^2\right)\left(\sum_{k=N}^M |b_k|^2\right)$$

Exercise: Given $\epsilon > 0$ show that there is an N so that for *any* M we have

$$\left(\sum_{k=N}^M |a_k|^2\right) < \epsilon \text{ and } \left(\sum_{k=N}^M |b_k|^2\right) < \epsilon$$

(Hint: Use the fact that the sum from $k = 1$ to infinity converges in each case.)

Exercise: Show that $\sum_{k=1}^\infty a_k b_k$ converges when a and b lie in ℓ_2.

We can thus define a pairing on ℓ_2 by the formula

$$\langle a, b\rangle = \sum_{k=1}^\infty a_k b_k$$

Exercise: Check that this is an inner product on ℓ_2.

We note that, along the way, we have shown that *any* linear functional on $\mathbb{R}^\infty$ that is continuous with respect to $\|\cdot\|_2$ is given by inner-product with a vector in ℓ_2. Since $\mathbb{R}^\infty$ is dense in ℓ_2, the same holds for a continuous linear functionals on ℓ_2. In other words, we have shown that $h : \ell_2 \to B(\ell_2, \mathbb{R})$ defined by

$$b \mapsto h_b \text{ where } h_b(a) = \langle a, b \rangle$$

is an isomorphism. In fact, as seen above, it also preserves the norm. This is the statement of Riesz representation theorem for ℓ_2. We will see a different proof in the next section.