Inner Product Spaces

In the initial part of this section we will work over the field $\mathbb{R}$ of real numbers.

An inner product space over $\mathbb{R}$ is a vector space over $\mathbb{R}$ together with an *inner product* $\langle , \rangle : V \times V \to \mathbb{R}$ such that:

1. (Bi-linearity) $\langle a \cdot v_1 + v_2, w\rangle = a\langle v_1, w\rangle + \langle v_2, w\rangle$.
2. (Symmetry) $\langle w, v\rangle = \langle v, w\rangle$.
3. (Positivity) $\langle v, v\rangle \geq 0$ and equality holds if and only if $v = 0$.

We can then define the norm $\|v\|$ by the formula $\|v\| = \sqrt{\langle v, v\rangle}$.

Exercise: Check that $2\langle v, w\rangle = \|v + w\|^2 - \|v\|^2 - \|w\|^2$.

Exercise: Deduce the "parallelogram law":

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

Exercise: Check that the above identity is *not* satisfied by the $\|\cdot\|_p$ norms on $\mathbb{R}^\infty$ for $p = 1$ or $p = \infty$ by finding vectors v and w which do not satisfy it.

Exercise: Deduce the identity:

$$\langle v, w\rangle = \frac{\|v + w\|^2 - \|v - w\|^2}{4}$$

In fact, one can show that if V is a normed vector space such that the norm *satisfies* the parallelogram law, then the above identity *defines* an inner product (i.e. that the above definition satisfies bi-linearity; symmetry and positivity are obvious!). (Try this as a starred exercise!)

Exercise: Check that the space $\mathbb{R}^n$ with the "usual" pairing:

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i$$

where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, defines an inner product space. Note that the resulting norm defined as above makes it a complete normed space.

Gram-Schmidt Orthogonalisation

Given an inner product space V. A sequence or collection $v^{(1)}, v^{(2)}, \ldots,$ of non-zero vectors in V is said to be *orthogonal* if, for all i and j so that $i \neq j$, we have $\langle v^{(i)}, v^{(j)}\rangle = 0$.

If, in addition to being orthogonal, each vector $v^{(i)}$ has norm 1, we say that this is an *orthonormal* collection or sequence of vectors.

Given an orthogonal collection $v^{(1)}, v^{(2)}, \ldots,$, we note that $\|v^{(i)}\| \neq 0$ for all i as the vectors are non-zero. It follows easily that if $w^{(i)} = v^{(i)}/\|v^{(i)}\|$, then $w^{(1)}, \ldots, w^{(n)}$ is an orthonormal collection.

Exercise: Check that an orthogonal collection of vectors is linearly independent.

If w is any vector and we have a *finite* orthonormal collection $v^{(1)}, \ldots, v^{(n)}$, then we can form the linear combination

$$w' = w - \left(\langle w, v^{(1)}\rangle \cdot v^{(1)} + \cdots + \langle w, v^{(n)}\rangle \cdot v^{(n)}\right)$$

Exercise: Check that w' satisfies $\langle w', v^{(k)}\rangle = 0$ for all $k = 1, \ldots, n$.

Geometrically, w' is the perpendicular "dropped" from w onto the linear subspace spanned by $v^{(1)}, \ldots, v^{(n)}$. We note that w' is non-zero if and only if w is not in this subspace. Equivalently, w' is linearly independent from the collection $v^{(1)}, \ldots, v^{(n)}$ if and only if w' is so.

Exercise: Given an orthonormal sequence $e^{(1)}, \ldots, e^{(n)}$ of vectors and v is a vector in the linear span of these vectors. Show that:

$$v = \langle v, e^{(1)}\rangle \cdot e^{(1)} + \cdots + \langle v, e^{(n)}\rangle \cdot e^{(n)}$$

The Gram-Schmidt process starts with any linearly independent sequence $v^{(1)}, \ldots$ and inductively creates an orthonormal sequence $w^{(1)}, \ldots$, such that for every n, the linear span of $v^{(1)}, \ldots, v^{(n)}$ is the same as the linear span of $w^{(1)}, \ldots, w^{(n)}$.

We first put $w^{(1)} = v^{(1)}/\|v^{(1)}\|$. Next, by induction the linear space spanned by $v^{(1)}, \ldots, v^{(n)}$ is the same as the linear space spanned by $w^{(1)}, \ldots, w^{(n)}$. Hence, $v^{(n+1)}$ does not belong to the latter space and the perpendicular w' from $v^{(n)}$ to this subspace is non-zero. So the following vector is well-defined

$$w^{(n+1)} = \frac{w'}{\|w'\|} = \frac{v^{(n+1)} - \left(\langle w, v^{(1)}\rangle v^{(1)} + \cdots + \langle w, v^{(n)}\rangle v^{(n)}\right)}{\text{norm of the numerator}}$$

We see that $w^{(n+1)}$ is a unit vector orthogonal to each $w^{(i)}$ for $i = 1, \ldots, n$. Continuing this for all n gives us the Gram-Schmidt process.

Finite Dimensional Spaces

Exercise: Given vectors v and w in the span of an orthonormal sequence $e^{(1)}, \ldots, e^{(n)}$. Show that

$$\langle v, w\rangle = \langle v, e^{(1)}\rangle\langle e^{(1)}, w\rangle + \cdots + \langle v, e^{(n)}\rangle\langle e^{(n)}, w\rangle$$

We thus see that the linear span of a *finite* orthonormal sequence $e^{(1)}, \ldots, e^{(n)}$ of vectors is isomorphic to $\mathbb{R}^n$ with the usual inner product via the map

$$v \mapsto \left(\langle v, e^{(1)}\rangle, \ldots, \langle v, e^{(n)}\rangle\right)$$

More generally, given *any* finite collection of vectors $v^{(1)}, \ldots, v^{(m)}$ in an inner product space, we can choose a maximal linearly independent subset $w^{(1)}, \ldots, w^{(n)}$ of them and apply the Gram-Schmidt process to them to obtain an orthonormal sequence $e^{(1)}, \ldots, e^{(n)}$ with the same linear span as $v^{(1)}, \ldots, v^{(n)}$. As seen above, this space is then isomorphic to $\mathbb{R}^n$. Thus:

Any geometric property of a finite collection of vectors in an inner product space over $\mathbb{R}$ is replicated as a property of a finite collection of vectors in $\mathbb{R}^n$.

This allows us to carry over all our calculations from "classical" linear algebra to the study of infinite dimensional linear spaces. For example, we have the identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

This gives us the inequality

$$(ac + db)^2 \leq (a^2 + b^2)(c^2 + d^2)$$

with equality if and only if $ad = bc$. Applying this to vectors $v = (a, b)$ and $w = (c, d)$ in $\mathbb{R}^2$ we have:

$$\langle v, w \rangle^2 \leq \|v\|^2 \|w\||^2$$

with equality if and only if v and w are linearly dependent. It follows that this statement is true in *any* inner product space over $\mathbb{R}$. With a view to later developments, this is usually written as follows:

Cauchy-Schwarz Inequality: Given a pair of vectors v and w in an inner product space, we have:

$$|\langle v, w \rangle| \leq \|v\| \|w\|$$

with equality if and only if v and w are linearly dependent.

Complex Numbers and Hermitian spaces

A vector space over the field of complex numbers can be, by restriction considered to be a vector space over the field of real numbers. Conversely, a vector space V over the field of real numbers, together with an endomorphism $j : V \to V$ so that j^2 acts as -1 can be considered as a vector space over the field of complex numbers as follows. For a complex number $z = x + \iota y$ and a vector v in V, we define

$$z \cdot v = x \cdot v + y \cdot j(v)$$

Exercise: Check that the above multiplication defines the structure of a complex vector space on V such that the underlying real vector space is the same as before.

Now suppose, in addition, that $\|\cdot\| : V \to \mathbb{R}$ is a norm on V. We then require that j is an isometry. In other words, $\|j(v)\| = \|v\|$. **Warning**: For a general norm this does not ensure that $\|z \cdot v\| = |z| \|v\|$!

Exercise: On the space $\mathcal{C}(\mathbb{R})$ of Cauchy sequences in $\mathbb{R}$ define the operation $j(v)$ by $j(v)_{2k} = v_{2k-1}$ and $j(v)_{2k-1} = -v_{2k}$ for $k = 1, \ldots$. Check that j is an isometry and $j^2 = -1$. Further, find a vector w so that $\|(1+j)(w)\| = 2\|w\|$, whereas $|1+\iota| = \sqrt{2}$.

Now suppose, in addition, that $\langle , \rangle : V \times V \to \mathbb{R}$ is an inner product on V and that $\|\cdot\|$ is the associated norm.

Exercise: Check that $\langle j(v), j(w) \rangle = \langle v, w \rangle$ for all v and w in V. (Hint: Use the formula for the inner product in terms of the norm.)

As a consequence, we can deduce the property that j is a "rotation by 90 degrees".

Exercise: Check that $\langle j(v), v \rangle = 0$. (Hint: Apply j to both sides of the inner product and use the previous exercise.)

We can then use this to deduce the following:

Exercise: Given a complex number z and a vector v in V:

$$\langle z \cdot v, z \cdot v \rangle = |z|^2 \langle v, v \rangle$$

In particular, this shows that, for an inner product space, the condition that j is an isometry is enough to ensure that the norm is also a norm over the field of complex numbers.

A Hermitian inner product on a vector space V over the field of complex numbers $\mathbb{C}$ is a pairing $\{ , \} : V \times V \to \mathbb{C}$ so that:

1. (Bi-linearity) $\{z \cdot v_1 + v_2, w\} = z\{v_1, w\} + \{v_2, w\}$ for every complex number z.
2. (Hermitian Property) $\{w, v\} = \overline{\{v, w\}}$, where $z \mapsto \overline{z}$ denotes complex conjugation as usual.
3. (Positivity) $\{v, v\} \geq 0$ and equality holds if and only if $v = 0$.

Exercise: Check that $\{v, z \cdot w\} = \overline{z}\{v, w\}$.

Given such pairing, one can define $\langle v, w \rangle$ to be the *real part* of $\{v, w\}$. This gives a pairing $V \times V \to \mathbb{R}$.

Exercise: Check that this pairing is an inner product for the underlying real vector space V.

Exercise: Check that $\langle \iota \cdot v, \iota \cdot w \rangle = \langle v, w \rangle$.

Exercise: Check that the imaginary part of $\{v, w\}$ is the same as the real part of $-\{\iota \cdot v, w\}$.

Note that the real part of $-\{\iota \cdot v, w\}$ is, by definition, the same as $-\langle \iota \cdot v, w \rangle$.

Exercise: Check that $\{v, w\} = \langle v, w \rangle - \iota \langle \iota v, w \rangle$. (Hint: Compare real and imaginary parts.)

The above exercises can now be combined into a relation between Hermitian inner product spaces over $\mathbb{C}$ and inner product spaces over $\mathbb{R}$.

Exercise: Given a Hermitian inner product $\{,\}$ on a vector space V over $\mathbb{C}$, the pairing $\langle v, w\rangle = \Re\{v, w\}$ (real part) is an inner product of V as a vector space over $\mathbb{R}$. Moreover, $j : v \mapsto \iota \cdot v$ is an $\mathbb{R}$-linear map $j : V \to V$ which is an isometry for this inner product space.

Conversely, we have:

Exercise: Given an inner product space V and an isometry $j : V \to V$ with $j^2 = -1$, there is a natural structure of a $\mathbb{C}$-vector space on V defined by $z \cdot v = \Re(z)v + \Im(z)j(v)$. Moreover, $\{v, w\} = \langle v, w\rangle - \iota\langle j(v), w\rangle$ defines a Hermitian inner product on this space. (Here, $(\Re(z), \Im(z))$ denote the real and imaginary parts of a complex number.)

Moreover, the norm of a vector is given by:

$$\|v\| = \sqrt{\{v, v\}} = \sqrt{\langle v, v\rangle}$$

Note that both $\{v, v\}$ and $\langle v, v\rangle$ are non-negative real numbers and thus have a unique non-negative square root.

Exercise: Use the Cauchy-Schwarz inequality for $\langle,\rangle$ to prove that for any two vectors v and v in a Hermitian inner product space V, we have:

$$|\{v, w\}| \leq \|v\|\|w\|$$

Due to the above discussion, we will not distinguish too much between the study of inner product over real numbers and Hermitian inner product spaces over complex numbers. The only thing to remember is that in the latter case we have a real isometry j with $j^1 = -1$ and that the real inner product is the real part of a Hermitian inner product.