1 Normed Spaces

A *norm* on a vector space V is a function $\|\cdot\| : V \to \mathbb{R}^{\geq 0}$ which satisfies:

1. (Definiteness) $\|a\| = 0$ if and only if $a = 0$.
2. (Scaling) $\|z \cdot a\| = |z| \cdot \|a\|$ for all z in $\mathbb{C}$ and for all a in V.
3. (Triangle Inequality) $\|a + b\| \leq \|a\| + \|b\|$ for all a and b in V.

Exercise: Note that we can define a metric on V by putting $d(a, b) = \|a - b\|$. Show that the multiplication $\mathbb{C} \times V \to V$ and addition $V \times V \to V$ are continuous with respect to this metric.

We will study various spaces obtained from various norms on $\mathbb{C}^\infty$. We have already seen $\mathcal{C}$, $\mathcal{C}_0$, ℓ_1 and ℓ_∞.

Note that given a norm on V, we can define $d(a, b) = \|a - b\|$.

Exercise: Show that d defines a metric on V.

As a result, we can also define a topology on V using the metric.

Exercise: Check that a set of the form $B^\circ_\epsilon(0) = \{a : \|a\| < \epsilon\}$ is an open set containing 0.

The set $B^\circ_\epsilon(0)$ is called an *open ball* around 0 in V of radius ϵ.

Exercise: Check that *any* open neighbourhood of 0 contains a set of the form $B^\circ_\epsilon = \{a : \|a\| \leq \epsilon\}$ for a suitable ϵ.

Exercise: Check that the closure of $B^\circ_\epsilon(0)$ is the set $B_\epsilon(0) = \{a : \|a\| \leq \epsilon\}$.

The set $B_\epsilon(0)$ is called a *closed ball* around 0 in V of radius ϵ.

1.1 Continuous Linear Maps

Given normed linear spaces V and W, we can study continuous linear maps between them. By definition, such a map $L : V \to W$ is linear *and* continuous. The continuity of L at 0 means that there is an $\delta > 0$ so that if x satisfies $d_V(x, 0) < \delta$, then $d_W(L(x), 0) < 1$ (here we are using the subscript to denote the space of the metric and norm.

Exercise: Check that $\|L(x)\|_W < (1/\delta)\|x\|_V$ for all x in V.

Conversely, suppose that $L : V \to W$ is a linear map and $r > 0$ a positive real number so that $\|L(x)\|_W \leq r\|x\|_V$ for all x in V. Such a linear map is called a *bounded* linear map.

Exercise: Check that $d_W(L(x), L(y)) \leq r d_V(x, y)$ for all x and y in V.

The above condition is sometimes called the Lipschitz condition on the map L.

Exercise: Check that a map between metric spaces that satisfies the Lipschitz condition is continuous.

As a result we see that a map between normed spaces is continuous if and only it is Lipschitz continuous. Moreover, the latter condition is equivalent to the condition that there is an $r > 0$ so that $\|L(x)\|_W < r\|x\|_V$ for all x in V. In other words, the notions of bounded linear maps and continuous linear maps coincide.

We define the *norm* of a continuous linear map $L : V \to W$ by

$$\|L\| = \sup\{\|L(x)\| : \|x\| \leq 1\}$$

We note that this exists since the right-hand side is bounded as proved above.

Exercise: Check that $\|L(x)\| \leq \|L\| \cdot \|x\|$ for all x in V.

Given a linear map $L : V \to W$, and a complex number z, it is clear that $zL : V \to W$ defined by $(zL)(x) = z \cdot L(x)$ is also a linear map. Moreover:

Exercise: Check that if L is continuous, then $\|z \cdot L\| = |z| \cdot \|L\|$, so that zL is continuous as well.

Similarly, if $L : V \to W$ and $M : V \to W$ are continuous linear maps then we have:

Exercise: Check that for every x in V we have

$$\|L(x) + M(x)\| \leq (\|L\| + \|M\|) \cdot \|x\|$$

It follows that if we define $(L + M)(x) = L(x) + M(x)$, then $L + M : V \to W$ is also a continuous linear map and $\|L + M\| \leq \|L\| + \|M\|$. The above two statements thus show that the collection

$$B(V, W) = \{L : V \to W \text{ such that } L \text{ is continuous linear } \}$$

is also a normed linear space.

Given a continuous linear map $L : V \to W$ and another $M : W \to U$, we have

$$\|L(x)\| \leq \|L\| \cdot \|x\| \ \forall x \in V$$

and

$$\|M(y)\| \leq \|M\| \cdot \|y\| \ \forall y \in W$$

It follows that

$$\|(M \circ L)(x)\| \leq \|M\| \cdot \|L\| \cdot \|x\| \ \forall x \in V$$

Thus, we see that $M \circ L : V \to U$ is a continuous linear map and $\|M \circ L\| \leq \|M\| \cdot \|L\|$.

1.2 Continuous linear functionals

A special important case of $B(V,W)$ is when $W = \mathbb{C}$ is just the standard 1-dimensional vector space with the usual norm. We use the notation V^* for the space $B(V,\mathbb{C})$ and call it the *dual* space of V; note that it is (in general) *strictly* smaller than the space of *all* (not necessarily continuous) linear functionals. We have already seen examples of this in the context of the space of sequences.

Given a linear functional $f : V \to \mathbb{C}$ we can break it into its real and imaginary parts $f = g + \iota \cdot h$, where $g, h : V \to \mathbb{R}$ are *real* linear functionals. We note that

$$f(\iota \cdot v) = \iota \cdot f(v) = \iota \cdot g(v) - h(v)$$

On the other hand,

$$f(\iota \cdot v) = g(\iota \cdot v) + \iota \cdot h(\iota v)$$

This shows that g and h determine each other by the formula

$$g(v) = h(\iota \cdot v) \text{ and } h(v) = -g(\iota \cdot v)$$

Conversely, given a linear functional $g : V \to \mathbb{R}$, we can *define* $f : V \to \mathbb{C}$ by the formula $f(v) = g(v) - \iota \cdot g(\iota \cdot v)$. (Note that since V is a *complex* vector space, the notion of ιv makes sense for any vector v in V.)

Exercise: Check that $f : V \to \mathbb{C}$ as defined above is a $\mathbb{C}$-linear functional on V.

We can define the norm $\|g\| = \sup\{|g(v)| : \|v\| \leq 1\}$. Since $|g(v)| \leq |f(v)|$ we see that $\|g\| \leq \|f\|$.

Exercise: Given $\epsilon > 0$, there is a vector v such that $\|v\| \leq 1$ and $|f(v)| > \|f\| - \epsilon$.

In particular, by taking $\epsilon < \|f\|$ we have $f(v) \neq 0$. We then put $w = \frac{|f(v)|}{f(v)} v$.

Exercise: Check that $f(w) = |f(v)|$ and $\|w\| = \|v\|$.

It follows that $g(w) = |f(v)|$ and thus, $\|g\| \geq |g(w)| > \|f\| - \epsilon$. Since we have this for all sufficiently small ϵ, it follows that $\|g\| \geq \|f\|$. In other words, we see that $\|g\| = \|f\|$.

1.3 Hahn-Banach Theorem

So far, we have talked about the space $B(V,W)$, but we have not shown that it is non-zero! If x is a non-zero vector in W, we can define a linear map $e_x : \mathbb{C} \to W$ be defining $z \mapsto zx$.

Exercise: Show that e_x is a linear map and $\|e_x\| = \|x\|$.

It follows that $B(\mathbb{C},W)$ can be identified with W and is non-zero if W is non-zero. To get a non-zero element of $B(V,W)$ for a general V, we can try to first create a non-zero element of $B(V, \mathbb{C})$ and then compose with the e_x. So the problem is to find a non-zero element of V^*.

If V is one dimensional, and $x \in V$ is a non-zero element, we have a continuous linear map $e_x : \mathbb{C} \to V$ given by $z \mapsto zx$ as above.

Exercise: The map e_x is 1-1 and onto when V is one dimensional. Moreover, in this case, its inverse is a continuous linear functional $f : V \to \mathbb{C}$ with $\|f\| = 1/\|x\|$ and $f(x) = 1$.

Replacing f by its multiple $\|x\|f$, we see that in case V is one dimensional there is a continuous linear functional $f : V \to \mathbb{C}$ so that $f(x) = \|x\|$ and $\|f\| = 1$.

More generally, given any normed space V and a non-zero vector x we can produce a linear functional $f : W = \mathbb{C}x \to \mathbb{C}$ so that $\|f\| = 1$. The Hahn-Banach theorem stated below allows us to extend this to all of V.

Hahn-Banach Theorem: Given a normed space V, a subspace W of V and a linear functional $f : W \to \mathbb{C}$ such that $\|f\| = 1$ as a linear functional on W. Then there is a linear functional $g : V \to \mathbb{C}$ such that $\|g\| = 1$ and g restricts to f on W.

This result is proved using Zorn's lemma as follows. Consider the collection $\mathcal{F}$ of pairs (U, h) where U is a subspace of V containing W, and $h : U \to \mathbb{C}$ is a linear functional such that $\|h\| = 1$ and h restricts to f on W. This has a partial order by declaring $(U, h) \leq (U', h')$ if $U \subset U'$ and h' restricts to h on U. Given any totally ordered chain $\{(U_i, h_i)\}$ in $\mathcal{F}$, we form $U = \cup_i U_i$ and define $h : U \to \mathbb{C}$ by $h(u) = h_i(u)$ if $u \in U_i$.

Exercise: Show that U is a subspace of V and that h is a linear functional on U with $\|h\| = 1$.

It follows that (U, h) bounds this totally ordered chain. By Zorn's lemma, there is a maximal element (U, h) in $\mathcal{F}$. We will prove by contradiction that $U = V$. To do this, we need the following extension argument which shows that if U is a proper subspace of V, then h can be extended to a larger subspace keeping the norm as 1. This contradicts the maximality of (U, h), thereby proving $U = V$ by contradiction as required.

1.3.1 Extending a linear functional

Given a normed linear space V and a subspace W, suppose we have a continuous linear functional $f : W \to \mathbb{C}$ with $\|f\| = 1$. What we want to do is to extend it to a larger subspace of V.

Let us therefore assume that $x \in V \setminus W$ is a vector and consider the subspace $U = W + \mathbb{C} \cdot x$. We want to define a continuous linear functional $g : U \to \mathbb{C}$ so that g restricts to f on $W \subset U$ and $\|g\| = 1$.

Exercise: Show that every element of U can be *uniquely* written in the form $w + t \cdot x$ for w in W and t a complex number.

Exercise: Choose *any* complex number z and define $h_z : U \to \mathbb{C}$ by the formula $h_z(w + t \cdot x) = f(w) + t \cdot z$ using the expression above. Prove that h_z is linear and h_z restricts to f on W.

However, we have not proved that h_z is continuous. So the only task we have is to choose z suitably so that $\|h_z\| = 1$. We will now see a way to do this (in a somewhat convoluted way) using the real part of f as described earlier.

Exercise: Show that every element of U can be *uniquely* written in the form $w + p \cdot x + q \cdot (\iota \cdot x)$ for w in W and p, q *real* numbers.

Let $f_0 : W \to \mathbb{R}$ be the real part of f. We will extend it to $g_0 : U \to \mathbb{R}$ in two steps by extending it to one more dimension at a time.

Given a, b in W, we have the inequality:

$$f_0(a) + f_0(b) = f_0(a + b) \leq \|a + b\| \leq \|(a - x) + (x + b)\| \leq \|a - x\| + \|x + b\|$$

If we put $\phi(a) = f_0(a) - \|a - x\|$ for $b \in W$, then:

$$\phi(a) = f_0(a) - \|x - a\| \leq \|b + x\| - f_0(b) = -\phi(-b)$$

This is true for all a and b in W, Fixing b, it follows that $r = \sup_{a \in W} \phi(a)$ satisfies $r \leq -\phi(-b)$. Since this is true for all b, it follows that:

$$f_0(a) - \|a - x\| \leq r \leq \|b + x\| - f_0(b) \ \forall a, b \in W$$

We now define $k_0(w + t \cdot x) = f_0(w) + t \cdot r$. This defines a linear functional on $W + \mathbb{R} \cdot x$. If $t > 0$ and w is an element of W, then we have:

$$k_0(w + t \cdot x) = t\,(f_0(w/t) + r) \leq t\|w/t + x\| = \|w + t \cdot x\|$$

We similarly check that:

$$k_0(w - t \cdot x) \leq \|w - t \cdot x\|$$

Exercise: Combine the above calculations to show that $|k_0(w+t\cdot x)| \leq \|w+t\cdot x\|$ for all w in W and all real numbers t. Deduce that $\|k_0\| = 1$. (Note that $\|f_0\| = 1$.)

It follows that $k_0 : W + \mathbb{R} \cdot x \to \mathbb{R}$ is a $\mathbb{R}$-linear functional which extends f_0 and has norm 1.

We can now *repeat* the above argument with $W_1 = W + \mathbb{R} \cdot x$ in place of W, $\iota \cdot x$ in place of x and k_0 in place of f_0, to produce $g_0 : W_1 + \mathbb{R} \cdot (\iota \cdot x) \to \mathbb{R}$ which is *real* linear, extends k_0 and has norm 1.

As seen above, $U = W_1 + \mathbb{R}$, so we have a real linear functional $g_0 : U \to \mathbb{R}$ that extends f_0 which is the real part of f and so that $\|g_0\| = 1$. We have seen that $g(u) = g_0(u) - \iota g(\iota \cdot u)$ is then a complex linear functional such that $\|g\| = \|g_0\|$. Hence we have the required extension of f to U.

1.4 Semi-norms and Closed Subspaces

A function $p : V \to \mathbb{R}^{\geq 0}$ is called a semi-norm if it satisfies:

1. (Linearity) $p(z \cdot a) = |z| \cdot p(a)$ for all z in $\mathbb{C}$ and for all a in V.
2. (Triangle Inequality) $p(a + b) \leq p(a) + p(b)$ for all a and b in V.

In other words, it does not have the additional property $p(a) = 0 \implies a = 0$ which a norm has. Consider the set $N(p) = \{v : p(v) = 0\}$.

Exercise: Check that $N(p)$ is stable under scalar multiplication and addition of vectors in V.

In other words, $N(p)$ is a subspace. In addition,

Exercise: Given v in V and w in $N(p)$ check that $p(v + w) = p(v)$.

As a consequence p induces a function $q : V/W \to \mathbb{R}^{\geq 0}$ on the quotient linear space V/W.

Exercise: Check that q defines a norm on V/W.

Conversely, it is easy to see that a norm on V/W gives, by composition with the natural linear map $V \to V/W$, a semi-norm on V.

In the above discussion, we have not put any relation between the semi-norm on V and *any* topology on V, we only need it to be a vector space.

Given a normed space V and a subspace W, we can define a function:

$$p_W(v) = \inf\{\|v + w\| : w \in W\}$$

We note that if t is a complex number with $t \neq 0$, then $\|t \cdot v + w\| = |t| \cdot \|v + w/t\|$

Exercise: Check that $p_W(t \cdot v) = |t| p_W(v)$ for all complex numbers t and all v in V.

Secondly, we note that

$$p_W(v) = \inf\{\|v + w_1 + w_2\| : w_1, w_2 \in W\}$$

Further, by the triangle inequality for $\|\cdot\|$, we have

$$\|v_1 + v_2 + w_1 + w_2\| \leq \|v_1 + w_1\| + \|v_2 + w_2\|$$

Exercise: Check that $p_W(v_1 + v_2) \leq p_W(v_1) + p_W(v_2)$. (Be careful with your argument since

$$\inf\{1/n + (1 - 1/n) : n \geq 2\} \neq \inf\{1/n : n \geq 2\} + \inf\{1 - 1/n : n \geq 2\}$$

In other words, p_W is a semi-norm. It follows that:

1. $N(p_W)$ is a linear subspace of V. It is obvious that it contains W.

2. p_W induces a norm on $V/\left(N(p_W)\right)$.

Exercise: Given that $p_W(v) = 0$ show that for every positive integer n, there is a vector w_n in W such that $\|v - w_n\| < 1/n$.

It follows that w_n is a sequence of vectors converging to v. Thus, $p_W(v) = 0$ implies that v lies in the closure of W. Conversely, if v lies in the closure of v, there is a sequence of vectors w_n in W as above. It follows that $p_W(v) < 1/n$ for all positive integers n, so $p_W(v) = 0$. Hence, $N(p_W)$ is the closure of W. So we have proved that the closure of W is a subspace and identified this with $N(p_W)$. In particular, if W is closed, we see that $N(p_W) = W$.

We have a natural linear map $V \to V/N(p_W)$. As seen above p_W induces a norm on p_W. Moreover, we have $p_W(v) = \inf_{w \in W} \|v + w\| \leq \|v\|$. So this linear map is continuous with respect to this norm.

More generally, if p is a semi-norm on V and is *continuous* with respect to the norm $\|\cdot\|$ on V, then there is an $\delta > 0$ so that if v satisfies $\|v\| < \delta$, then $p(v) < 1$.

Exercise: In this case show that $p(v) \leq (1/\delta)\|v\|$ for *all* v in V.

It follows that the natural morphism $V \to V/N(p)$ is continuous and that $N(p)$ is closed in V with respect to the norm topology on V. As seen above, $p(v+n) = p(v)$ for all n in $N(p)$. It follows that

Exercise: $p(v) \leq (1/\delta)p_{N(p)}(v)$.

We thus see that there are two norms on $V/N(p)$, the one given by p and the one given by $p_{N(p)}$. The second norm dominates the first one in the following sense.

1.4.1 Stronger or Finer norms

If V has two norms, $\|\cdot\|_a$ and $\|\cdot\|_b$, then we can ask whether these norms can be compared. Let V_a denote the normed space with the a norm and V_b denote the normed space with the b norm. If any open set in V_b is also an open set in V_a, then the identity map $V_a \to V_b$ is continuous. As seen above, this means that there is a constant $r > 0$ so that

$$\|x\|_b \leq r\|x\|_a \ \forall x \in V$$

It is therefore natural to say that the b norm is *dominated* by the a norm or that it is *weaker* than the a norm in this case.

Exercise: Suppose there is a constant $r > 0$ so that the above inequality is satisfied for all x in V. Show that the identity map $V_a \to V_b$ is continuous.

Now, it may happen that each norm is weaker than the other! In other words, it may happen that there are constants $r > 0$ and $s > 0$ so that

$$s\|x\|_a \leq \|x\|_b \leq r\|x\|_a \ \forall x \in V$$

In this case, we say that the norms are *equivalent*. Note that in this case the underlying topology is the same.

1.4.2 Finite supplements

Let W be a subspace of a normed linear space V and v be a vector in V that is not contained in W. We then have a subspace $U = W + \mathbb{C}v$ of V which contains W. There are two possible cases to consider.

In the first case, the closure of W in U is U. This is the same as saying the v lies in the closure of W since if w_n converges to v then $z \cdot w_n$ converges to $z \cdot v$. This can also be characterised as the case where $p_W(v) = 0$ as seen above.

In the second case, $p_W(v) > 0$ and p_W induces a norm on U/W. The latter is a 1-dimensional space and so all norms are equivalent. In fact:

Exercise: Show that $\|w + z \cdot v\| \geq |z| p_W(v)$ for all complex numbers z.

It follows that $f : U \to \mathbb{C}$ defined by $f(w + z \cdot v) = z$ is a continuous linear function with norm $1/p_W(v)$. Note that that $g : U \to W$ given by $g(w+z\cdot v) = w$ can be given by the formula $g(u) = u - f(u)v$ and is therefore continuous also. In fact:

Exercise: Show that $\|w\| \leq (1 + \|v\|/p_W(v))\|w + z \cdot v\|$.

We can combine these inequalities to get:

Exercise: Show that

$$\|w\| + |z| \leq (1 + \|v\|/p_W(v) + 1/p_W(v)) \cdot \|w + z \cdot v\|$$

We note that $\|w + z \cdot v\| \leq \|w\| + |z| \cdot \|v\|$. Hence:

Exercise: Show that $\|w + z \cdot v\| \leq \max\{1, \|v\|\} (\|w\| + |z|)$

In other words, the given norm on U is equivalent to the norm defined by

$$\|w + z \cdot v\|_1 = \|w\| + |z|$$

We can generalise this inductively as follows. Let F be a subspace of V which is finite dimensional and $F \cap W = \{0\}$. If W is closed in $U = W + F$, then the norm on U (restricted from V) is equivalent to the norm

$$\|w + f\|_1 = \|w\| + \|f\|_0$$

where $\|f\|_0$ is *any* norm on the finite dimensional space F. In particular, all norms on a finite dimensional space are equivalent. Moreover, the condition that W is closed in U is equivalent to the condition that p_W gives a norm on F.

1.5 Completion

Given a norm on V, we can "complete" V with respect to the associated metric. In the following sequence of definitions and exercises, we will show how this is done using the ideas developed above.

As usual we define a *Cauchy* sequence (v_n) as a sequence of vectors v_n in V such that for any $\epsilon > 0$, there is an $N(\epsilon)$ so that, if $n, m > N(\epsilon)$, then $\|v_n - v_m\| < \epsilon$.

Exercise: If (v_n) and (b_n) are Cauchy sequences, then show that $(v_n + b_n)$ is also a Cauchy sequence.

Exercise: If (v_n) is a Cauchy sequence and z is a complex number then show that $(z \cdot v_n)$ is also a Cauchy sequence.

Exercise: Show that Cauchy sequences in V also form a vector space $C(V)$ in a natural way.

Exercise: Show that the following limit exists for a Cauchy sequence (v_n):

$$\lim_{n\to\infty} \|v_n\|$$

We then use this to define a function $p : C(V) \to \mathbb{R}^{\geq 0}$ by

$$p((v_n)) = \lim_{n\to\infty} \|v_n\|$$

Exercise: Check that p is a semi-norm on $C(V)$.

Let $C_0(V)$ be the space $N(p)$ considered above. In other words,

$$C_0(V) = \{(v_n) : \lim_{n\to\infty} \|v_n\| = 0\}$$

Exercise: Check that $C_0(V)$ is exactly the space of all sequences in V that converge to 0.

We see that $C(V)/C_0(V)$ becomes a normed linear space with the norm induced by p. There is a natural map $V \to C(V)$ given by sending a vector w to the constant sequence with $v_n = w$ for all n. This allows us to think of V as a subspace of $C(V)$. We note that $p(w) = \|w\|$ with respect to this inclusion. Since $\|\cdot\|$ is a norm on V, we see that $V \to C(V)/C_0(V)$ is an inclusion so that the norm on the latter space given by p gives existing norm on V.

The main result is that $C(V)/C_0(V)$ is complete as a metric space. To aid this, let us define a norm on $C(V)$ by

$$\|(v_n)\| = \sup_n \|v_n\|$$

Exercise: Check that this defines a norm on $C(V)$.

Secondly we note that $p((v_n)) \leq \|(v_n)\|$ and deduce that p is continuous with respect to this norm. It follows that the map $C(V) \to C(V)/C_0(V)$ is continuous.

The proof that $C(V)/C_0(V)$ is complete and that V is dense in it now follows a familiar line of argument used to prove that $\mathcal{C}_0$ is complete. It is left to the reader!