Spaces of Sequences

An important class of topological vector spaces which we will now study is the class of spaces of sequences. This will also give us an opportunity to revise some fundamental ideas from analysis.

To fix notions, we will work with vector spaces over the field $\mathbb{C}$ of complex numbers.

We begin with the vector space $\mathbb{C}^\infty$ consisting of all sequences that are *eventually* 0. In other words, this is the space of sequences $(a_n)_{n=1}^\infty$ such that there is an N so that $a_n = 0$ for all $n > N$; note that the N may be different for different elements of $\mathbb{C}^\infty$. For each k, consider the vector $e^{(k)}$ as the sequence $(\delta_{k,n})_{n=1}^\infty$; in other words, only the k-th entry of $e^{(k)}$ is 1 and all others are 0. Then $\mathbb{C}$ is the vector space consisting of linear combinations (which by definition are finite!) of the countable basis $\{e^{(k)}\}_{k=1}^\infty$.

Exercise: Show that every vector in $\mathbb{C}^\infty$ is a unique finite linear combination of the vectors $e^{(k)}$ for $k = 1, 2, \ldots$.

In order to put a topology on this space, we need to define a notion of distance. Given any two vectors $a = (a_n)$ and $b = (b_n)$ in $\mathbb{C}$, there is a common N so that $a_n = 0 = b_n$ for all $n > N$. In other words, a and b lie in the finite dimensional subspace $\mathbb{C}^N$ of $\mathbb{C}^\infty$ that consist of sequences that are 0 beyond the N-th entry. It then makes sense to take the distance between a and b to be the distance in this finite dimensional subspace $\mathbb{C}^N$. *However*, there are many such notions of distance and (as we shall see later) and that *will* make a difference to the space we will construct by "completion of the metric".

Norms

A *norm* on a vector space V is a function $||\cdot|| : V \to \mathbb{R}^{\geq 0}$ which satisfies:

1. (Positivity) $||a|| = 0$ if and only if $a = 0$.
2. (Linearity) $||z \cdot a|| = |z| \cdot ||a||$ for all z in $\mathbb{C}$ and for all a in V.
3. (Triangle Inequality) $||a + b|| \leq ||a|| + ||b||$ for all a and b in V.

Exercise: Note that we can define a metric on V by putting $d(a, b) = ||a - b||$. Show that the multiplication $\mathbb{C} \times V \to V$ and addition $V \times V \to V$ are continuous with respect to this metric.

Recall that *complete* metric space is one where any Cauchy sequence has a limit. Recall that a metric space has a *completion*. This means that the metric space (V, d) can be seen as a metric subspace of a complete metric space W.

In a later section we will see how this can be done for a normed space in particular. For the moment, it is worth remarking that we will study various spaces obtained as completions associated with various norms on $\mathbb{C}^\infty$.

The simplest norm on $\mathbb{C}^\infty$ is defined by defining it on $a = (a_n)$ by the formula $||a||_\infty = \sup_n |a_n|$. Note that since $a_n = 0$ for $n >> 0$, this is the same as $\max_n |a_n|$ in this particular case.

The space $\mathcal{C}$ of convergent sequences

The space $\mathbb{C}^\infty$ together with the norm $||\cdot||_\infty$ is often denoted as $\mathcal{C}_{00}$. This notation is chosen because of two (larger) spaces $\mathcal{C}_0$ and $\mathcal{C}$ which we will study below.

The space $\mathcal{C}$ is the space of convergent sequences of complex numbers. Since complex numbers form a complete metric space, we can also think of $\mathcal{C}$ as the space of all Cauchy sequences of complex numbers.

Exercise: Show that $\mathcal{C}$ is closed under addition of sequences and multiplication of a sequence by a fixed scalar. In other words, show that $\mathcal{C}$ is a vector space.

Exercise: For a convergent sequence (a_n) of complex numbers show that $\sup_n |a_n|$ exists. Note that this may be *different* from $\lim_n |a_n|$ which also exists by providing an example where the two are different.

We can thus define $||(a_n)||_\infty = \sup_n |a_n|$ on the space $\mathcal{C}$.

Exercise: Show that this defines a norm on $\mathcal{C}$.

Hence, there is a metric on $\mathcal{C}$ defined by $d(a, b) = ||a - b||_\infty$ as above.

Exercise: Show that a sequence $\{a^{(n)}\}$ is a Cauchy sequence in this metric if and only if, for every $\epsilon > 0$, there is an N so that $||a^{(n)} - a^{(m)}||_\infty < \epsilon$ for $n, m \geq N$.

To do the above exercise as well as the following exercises, it is useful to visualise the $a^{(n)}$ as the n-th row of an infinite by infinite matrix A.

Note that the k-th terms of this sequence (the columns of the matrix A) satisfy

$$|a_k^{(n)} - a_k^{(m)}| \leq ||a^{(n)} - a^{(m)}||$$

Using this we conclude that, for each k, $\{a_k^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence of complex numbers. Since complex numbers form a complete metric space, this sequence converges to a complex number b_k.

We claim that this sequence $b = (b_k)$ is in $\mathcal{C}$ and that $a^{(n)}$ converges to b in this metric space. To do this we will show that (b_k) is *itself* a Cauchy sequence of complex numbers.

Given $\epsilon > 0$, let N be such that $||a^{(n)} - a^{(m)}|| < \epsilon/3$ for $n, m \geq N$. Since $a^{(N)}$ (the N-th row of A) is a Cauchy sequence, there is a M so that $|a_k^{(N)} - a_l^{(N)}| < \epsilon/3$ for $k, l \geq M$.

Exercise: Show that $|a_k^{(n)} - a_l^{(m)}| < \epsilon$ for all $n, m \geq N$ and $k, l \geq M$. (Hint: In terms of the matrix A, note that for the N-th row and below, any two elements in the same column are within $\epsilon/3$ of each other. Moreover, any two elements in the N-th row beyond the M-th column are within $\epsilon/3$ of each other.)

Now, for a fixed k, the sequence $a_k^{(n)}$ (the k-th column of A) converges to b_k.

Exercise: Show that of $|a_k^{(N)} - b_k| \leq \epsilon/3$ for $k \geq M$. (Hint: b_k is the limit of $a_k^{(n)}$ for $n \geq N$.)

Exercise: As a consequence of the above reasoning, show that (b_k) is also a Cauchy sequence.

For each $i = 1, \ldots, M-1$, the sequence $a_i^{(n)}$ converges to b_i as n goes to infinity. So we can choose L_i so that $|b_i - a_i^{(n)}| < \epsilon$ for $n \geq L_i$. Let L be larger than all the L_i and also larger than N.

Exercise: Combine the above two exercises to show that $|b_k - a_k^{(n)}| \leq \epsilon$ for *all* k as long as $n \geq L$.

So we have shown that $||b - a^{(n)}||_\infty \leq \epsilon$ for $n > L$. Since $\epsilon > 0$ was arbitrary, we see that the sequence $a^{(n)}$ converges to b in $\mathcal{C}$ under the metric given by the norm $||\cdot||_\infty$. Hence $\mathcal{C}$ is a *complete* metric space.

The subspace $\mathcal{C}_0$ of $\mathcal{C}$ consists of sequences that converge to 0. Given a sequence $a = (a_k)$ in $\mathcal{C}$, let α be the limit of a_k. Then the sequence $(a_k - \alpha)$ is in $\mathcal{C}_0$. Let $e^{(0)}$ denote the sequence which is the "constant" sequence consisting of 1's. Then $a - \alpha \cdot e^{(0)}$ is the same as the sequence $(a_k - \alpha)$. In other words, we have written every element of $\mathcal{C}$ as a sum of an element of $\mathcal{C}_0$ and a multiple of $e^{(0)}$.

Exercise: If $\{a^{(n)}\}$ is a Cauchy sequence of elements of $\mathcal{C}_0$, then show that its limit in $\mathcal{C}$ as constructed above is also in $\mathcal{C}_0$.

In other words, $\mathcal{C}_0$ is a closed subspace of $\mathcal{C}$.

Given a sequence $a = (a_k)$ in $\mathcal{C}_0$, we can form the elements $a^{(n)}$ of $\mathcal{C}_{00}$ by defining $a_k^{(n)} = a_k$ for $k \leq n$ and $a_k^{(n)} = 0$ for $k \geq n$.

Exercise: Show that the sequence $a^{(n)}$ converges to a in $\mathcal{C}$.

It follows that $\mathcal{C}_{00}$ is dense in $\mathcal{C}_0$.

Counter Examples-1

In order to see the importance of the Cauchy property with respect to the $||\cdot||_\infty$ norm, let us consider the sequence of elements $a^{(n)}$ of $\mathcal{C}_{00}$ defined as follows:

$$a_k^{(n)} = \begin{cases} k & \text{for } k \leq n \\ 0 & \text{for } k > n \end{cases}$$

Note that for each fixed k, the sequences $(a_k^{(n)})_{n=1}^{\infty}$ are constant sequences with value k for $n \geq k$. In other words, the "columns" of our matrix are Cauchy sequences. Moreover, each $a^{(n)}$ is an element of $\mathcal{C}_{00}$. All the same, the limit sequence is the sequence of positive integers and that is *not* in $\mathcal{C}$!

In other words, it is not enough if the rows and columns of the matrix are Cauchy. The Cauchy property of the sequence of elements $a^{(n)}$ in $\mathcal{C}$ is essential to getting a common limit.

Continuous Linear Functionals

Since we are dealing with a *topological* vector space V, we need to study maps $f : V \to \mathbb{C}$ which are not only linear but also continuous. Such a map is called a "continuous linear functional" on V.

On the space $\mathcal{C}$ we have the obvious linear functionals corresponding to the "co-ordinates". The functional $f^{(k)}$ sends the sequence (a_n) to the complex number a_k.

Exercise: Check that $f^{(k)}$ is a continuous linear functional on $\mathcal{C}$ with respect to the $||\cdot||_\infty$ norm.

Note that this is the same as the statement proved above that the k-th co-ordinates of a Cauchy sequence in $\mathcal{C}$ form a Cauchy sequence of complex numbers.

The question we can ask ourselves is whether there are any other continuous linear functionals. Suppose that we are given a sequence $a^{(n)}$ in $\mathcal{C}$ that converges to b in $\mathcal{C}$. By definition of $\mathcal{C}$, for each fixed n, the sequence $a_k^{(n)}$ of complex numbers converges to a complex number α_n as k goes to infinity. Similarly, if b is the sequence (b_k), then the sequence b_k of complex numbers converges to a complex number β.

Exercise: Show that the sequence α_n converges to β.

Further, it is a standard fact that:

Exercise: If (a_k) converges to α and (b_k) converges to β, then (a_k+b_k) converges to $\alpha + \beta$ and, for any complex number z, the sequence (za_k) converges to $z\alpha$.

It follows that the map $g : \mathcal{C} \to \mathbb{C}$ that sends a sequence (a_n) to its limit α is a continuous linear functional on $\mathcal{C}$. Let us denote this as $f^{(0)} : \mathcal{C} \to \mathbb{C}$.

What other continuous linear functionals on $\mathcal{C}$ are there? Clearly, we can take *finite* linear combinations of linear functionals to get linear functionals. So we can form a finite sum like $\sum_{k \leq r} c_k f^{(k)}$ (with $f^{(k)}$ as above) to get a linear functional on $\mathcal{C}$. How about an infinite sum? Will it make sense. When does $\sum_{k=1}^{\infty} c_k a_k$ make sense for *all* sequences (a_k) in $\mathcal{C}$? Let us first see when it does *not* make sense!

Suppose the series $\sum_k c_k$ is *not* absolutely convergent; in other words $\sum_{k=1}^{\infty} |c_k|$ diverges to infinity. (Note that if $\sum_{k=1}^{N} |c_k|$ is bounded by a fixed bound for all N then, by Archimedes' principle, this series converges to a real number γ. This is the power of this principle!)

Exercise: Show that these is a sequence $n_0 = 0 < n_1 < n_2 < \cdots$ of integers so that $\sum_{k=n_{r-1}+1}^{n_r} |c_k| \geq r$ for $r \geq 1$.

We can then define a sequence a_k as follows:

$$a_k = \begin{cases} 0 & \text{if } c_k = 0 \\ \frac{1}{r}\frac{|c_k|}{c_k} & \text{if } c_k \neq 0 \text{ and } n_{r-1} < k \leq n_r \end{cases}$$

Exercise: With this definition, show that (a_k) lies in $\mathcal{C}_0$.

On the other hand, we see that

$$a_k \cdot c_k = \begin{cases} 0 & \text{if } c_k = 0 \\ \frac{|c_k|}{r} & \text{if } c_k \neq 0 \text{ and } n_{r-1} < k \leq n_r \end{cases}$$

Exercise: Show that $\sum_{k=n_{r-1}+1}^{n_r} a_k c_k \geq 1$. It follows that $\sum_{k=1}^{n_r} a_k c_k \geq r$.

In other words, the series $\sum a_k c_k$ does not converge. Thus, in order for such a sum to make sense for *all* elements (a_k) in $\mathcal{C}$ it is *necessary* that $\sum_k |c_k|$ be bounded; in other words, that the series $\sum_k c_k$ be *absolutely convergent*.

Given a continuous linear functional $f : \mathcal{C} \to \mathbb{C}$. Let us define $c_k = f(e^{(k)})$. Consider the linear functional f restricted $\mathcal{C}_{00} \oplus \mathbb{C} \cdot e^{(0)}$. It is clearly defined "formally" as $c_0 f^{(0)} + \sum_{k=1}^{\infty} c_k f^{(k)}$. On the other hand, as seen above this formal expression does not give a continuous linear functional on $\mathcal{C}$ unless $\sum_k c_k$ is absolutely convergent.

Exercise: Modify the above argument to show that the linear functional defined formally on $\mathcal{C}_{00}$ by the sum $\sum_{k=1}^{\infty} c_k f^{(k)}$ is not continuous in the $|| \cdot ||_\infty$ norm unless $\sum_k c_k$ is absolutely convergent. (Hint: The "truncated" sequences of the sequence (a_k) given above form a Cauchy sequence whose image is not Cauchy.)

On the other hand, if the above series *is* absolutely convergent, then:

Exercise: If $a = (a_k)$ is in $\mathcal{C}$, then $\sum_k |a_k c_k| \leq ||a||_\infty \sum_k |c_k|$.

So the formally defined linear functional on $\mathcal{C}_{00}$ extends by continuity to $\mathcal{C}$. Thus, the space of continuous linear functionals on $\mathcal{C}$ can be identified with the collection of functionals $c_0 f^{(0} + \sum_{k=1}^{\infty} c_k f^{(k)}$ where $\sum_k c_k$ is absolutely convergent.

Absolutely summable sequences

We define ℓ_1 to be the space of sequences (c_k) of complex numbers such that $\sum_k c_k$ is an absolutely convergent series; it is sometimes called the space of absolutely summable sequences. We define the norm on this space to be $||(c_k)||_1 = \sum_k |c_k|$.

Exercise: Check that $||\cdot||_1$ is a norm on the linear space ℓ_1.

As seen above, the space ℓ_1 can be identified with the space of continuous linear functionals on $\mathcal{C}_0$ (or $\mathcal{C}$). The linear space $\mathbb{C}^\infty$ is a subspace of ℓ_1 since finite sequences are clearly absolutely summable.

Conversely, given a sequence $c = (c_k)$ in ℓ_1, we can (as above) produce a sequence $c^{(n)}$ in $\mathbb{C}^\infty$ by defining

$$c_k^{(n)} = \begin{cases} c_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Exercise: For any $\epsilon > 0$, there is an N so that $\sum_{k>n} |c_k| < \epsilon$ for all $n > N$.

Using the above we can show that:

Exercise: The sequence $c^{(n)}$ in $\mathbb{C}^\infty$ converges to c with respect to the metric defined by the $||\cdot||_1$ norm on ℓ_1.

We have thus produced another norm $||\cdot||_1$ on the space $\mathbb{C}^\infty$ different from $||\cdot||_\infty$. However, at this point it is not clear whether the induced topology is different. To see that we would need to see that the completions $\mathcal{C}_0$ and ℓ_1 are different topological vector spaces. One way to see this is to study the space of continuous linear functionals on ℓ_1!

Bounded Sequences

Let $a = (a_k)$ be a sequence such that $\sup_k |a_k| < \infty$; in other words a is a bounded sequence. We define $||a||_\infty = \sup_k |a_k|$ as above.

Exercise: For an absolutely summable sequence $c = (c_k)$, show that $|\sum_k a_k c_k| \leq ||a||_\infty ||c||_1$. (Hint: The proof is exactly the same as the one used when a is a convergent sequence. That proof did not use the convergence of the sequence, merely its boundedness!)

Exercise: Check that the map $\ell_1 \to \mathbb{C}$ given by $c \mapsto \sum_k a_k c_k$ is a continuous linear functional on ℓ_1. Further note that $e^{(n)} \mapsto a_n$, so that if $a_n \neq 0$ for some n, then the linear functional is non-zero.

It follows that the space ℓ_∞ of bounded sequences of complex numbers is contained in the space of continuous linear functionals on ℓ_1.

Conversely, given a linear functional $f : \ell_1 \to \mathbb{C}$ we can define $a_n = f(e^{(n)})$. As before, it is clear that if $c = (c_k)$ is a sequence in $\mathbb{C}^\infty$, then $f(c)$ is the *finite* sum $\sum_k a_k c_k$. However, it is not clear whether this is true for *all* elements of ℓ_1. In fact, it is not clear that the sum $\sum_k a_k c_k$ even makes sense for all elements of ℓ_1.

Suppose that $a = (a_k)$ is *not* a bounded sequence. It follows that there is a sequence $n_1 < n_2 < \cdots$ of positive integers so that $|a_{n_r}| \geq r$. We now define a sequence $c = (c_k)$ as follows.

$$c_k = \begin{cases} 0 & \text{unless } k = n_r \text{ for some } r \\ \frac{1}{r^2}\frac{|a_{n_r}|}{a_{n_r}} & k = n_r \end{cases}$$

Exercise: Show that $c = (c_k)$ is an element of ℓ_1. (Hint: Note that $\sum_k |c_k| \leq \sum_r 1/r^2$.)

Moreover, we calculate

$$a_k \cdot c_k = \begin{cases} 0 & \text{unless } k = n_r \text{ for some } r \\ \frac{|a_{n_r}|}{r^2} & k = n_r \end{cases}$$

Exercise: Show that $\sum_k a_k c_k$ does not converge. (Hint: Note that this series dominates $\sum_r 1/r$.)

Exactly as above, we can use this to show that the functional $f : \mathbb{C}^\infty \to \mathbb{C}$ is continuous in the norm $||\cdot||_1$ only if $a = (a_n) = (f(e^{(n)}))$ is a bounded sequence. By the density argument above, it then follows that ℓ_∞ *is* the space of continuous linear functionals on ℓ_1.

By a number of different arguments, one can show that there is no embedding of $\mathbb{C}^\infty$ in ℓ_∞ which has dense image. It follows that ℓ_∞ is *not* isomorphic to ℓ_1 (which has such a dense embedding). It *therefore* follows that ℓ_1 is *not* isomorphic to $\mathcal{C}$ or $\mathcal{C}_0$.

A normed linear space is called *separable* is it contains a dense embedding of $\mathbb{C}^\infty$. The space ℓ_∞ is *not* a separable space; it is essentially the only such space that we will study. In the next section we will see that our primary objects of study can be identified with separable subspaces of ℓ_∞.