Expectation, Moments and Weak Law

1. Suppose that a discrete random variable X takes different values in the set $\{-2, -1, 0, 1, 2\}$ with the probabilities given as below:

Value	-1	0	1	2
Probability	1/10	2/5	1/5	1/10

   Answer the following questions:

   1. What is the value of $P(X = -2)$?
   2. What is the value of $E(X)$?
   3. Suppose Y is another random variable with distribution identical to X. What is $E(X + 1/(Y + 3))$
   4. What is the variance $\sigma^2(X)$?

2. Let X and Y be distinct real valued random variables.

   1. For all real numbers a and b check that $a^2E(X^2) + 2abE(XY) + b^2E(Y^2)$ is a non-negative number. (Hint: What is its relation with $E((aX + bY)^2)$?)
   2. Suppose A, B and C are real numbers such that $A + 2Bt + Ct^2 \geq 0$ for all real numbers t show that $AC \geq B^2$.
   3. Use the above two steps to conclude that $E(XY)^2 \leq E(X^2)E(Y^2)$.
   4. Check that $E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)$. (Hint: Expand!)
   5. Let us define $c(X, Y) = E(XY) - E(X)E(Y)$. Show that $c(X, Y)^2 \leq \sigma^2(X)\sigma^2(Y)$. (Hint: Apply steps 1-3 to $X - E(X)$ and $Y - E(Y)$.)
   6. Show that $c(X, Y) = 0$ if X and Y are independent.

   The number $c(X, Y)$ is called the covariance of X and Y. Assuming that $\sigma^2(X)\sigma^2(Y)$ is non-zero, the ratio $\rho(X, Y) = c(X, Y)/\sigma(X)\sigma(Y)$ is called the correlation of X and Y. It lies between -1 and 1.

3. Suppose that X and Y are independent random variables.

   1. Show that aX and bY are independent random variables for any non-zero constants a and b.
   2. Let $U = X + Y$ and $V = 2X - 3Y$. Calculate $E(U)$, $\sigma^2(U)$, $E(V)$ and $\sigma^2(V)$ in terms of $E(X)$, $\sigma^2(X)$, $E(Y)$ and $\sigma^2(Y)$.
   3. Let $W = 3X - 2Y = U + V$. Calculate $E(W)$ and $\sigma^2(W)$ in terms of $E(X)$, $E(Y)$, $\sigma^2(X)$ and $\sigma^2(Y)$.
   4. Check whether U and V are independent.

This exercise shows that independence is quite a different notion from what one might think since X and Y (which are independent) can be written linearly in terms of U and V which are not independent!

4. Given any two positive numbers a and b such that $b < a$ and $a^2 = k(a - b)$ for some positive integer k. Show that there is a random variable X such that $E(X) = a$ and $\sigma^2(X) = b$. (Hint: Consider a suitable Binomial random variable.)

5. Given any two positive numbers a and b such that $a < b$ and $a^2 = k(b - a)$ for some positive integer k. Show that there is a random variable X such that $E(X) = a$ and $\sigma^2(X) = b$. (Hint: Consider a suitable Negative Binomial random variable.)

   Note that, in fact, given any real number a and any positive number b, we can define a random variable with $E(X) = a$ and $\sigma^2(X) = b$. (The sign of a does not matter since $E(-X) = -a$. The tricky part is getting the case $|a| = b$!)

6. Suppose we flip a fair coin 100 times. Using Chebychev's inequality, estimate the probability that we see between 40 and 60 heads.