Solutions to Assignment 1

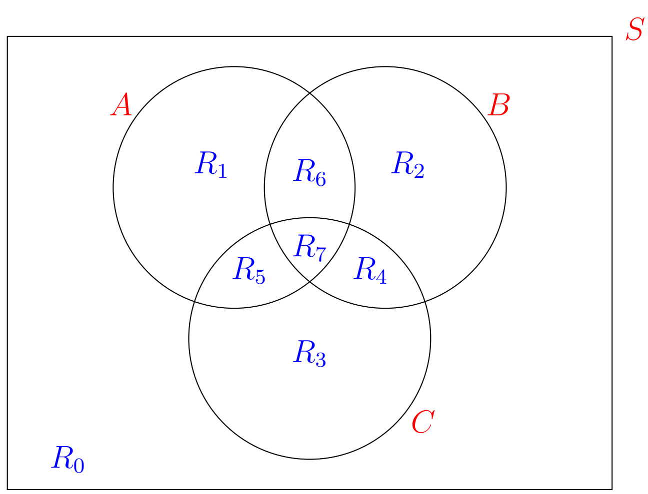
1. Set operations on subsets of a fixed set S are defined as follows:

   - $A \wedge B$ is the collection of all elements of S that are in A as well as B.
   - $A \vee B$ is the collection of all elements of S that are either in A or in B.
   - A^c is the collection of all elements of S that are not in A.
   - $A \setminus B$ is the collection of all elements of S that are in A but not in B.

   Which of the following are correct identities for subsets A, B, C of S? Provide examples if they are not correct or prove equality if they are correct.

$$
\begin{aligned}
(A \vee B) \wedge C &= (A \wedge C) \vee (B \wedge C) \\
(A \wedge B) \vee C &= (A \vee C) \wedge (B \vee C) \\
(A \setminus B) \wedge C &= (A \wedge C) \setminus (B \wedge C) \\
(A \setminus B) \vee C &= (A \vee C) \setminus B \\
A \setminus B &= A \wedge B^c \\
(A \wedge B)^c &= A^c \vee B^c
\end{aligned}
$$

Solution: We can draw a Venn diagram and note that there are 8 marked pieces which we have denoted as $R_0, \ldots, R_7$.



We note that:

- A consists of the pieces R_1, R_5, R_6 and R_7

- B consists of the pieces R_2, R_4, R_6 and R_7
- C consists of the pieces R_3, R_4, R_5 and R_7

For each identity, by examining which marked pieces are on the left-hand side and which are on the right-hand side we can decide whether the identity holds or not!

For example, for the proposed identity

$$(A \setminus B) \vee C \stackrel{?}{=} (A \vee C) \setminus B$$

the left hand side consists of R_1, R_3, R_4, R_5 and R_7. The right-hand side consists of R_1, R_3 and R_5. Hence the two sides are not equal.

2. For a subset A of S let $\chi_A : S \to \{0,1\}$ be the function whhich takes the value 1 on elements of A and 0 for elements of A^c. We define multiplication of elements of $\{0,1\}$ as usual and addition by

$$0 \oplus 0 = 0; 0 \oplus 1 = 1 \oplus 0 = 1 \text{ and } 1 \oplus 1 = 0$$

Write the following functions in terms of χ_A and χ_B.

- χ_{A^c}
- $\chi_{A \wedge B}$
- $\chi_{A \vee B}$
- $\chi_{A \setminus B}$

What subsets corresponds to the functions given below?

- $\chi_A \oplus \chi_B$
- $1 \oplus \chi_A$.
- $\chi_A \cdot (1 \oplus \chi_A)$.

Write the condition that A is contained in B in terms of the functions χ_A and χ_B.

Solution: We note that $1 \oplus \chi_A$ is $1 \oplus 1 = 0$ at all s in A and is $1 \oplus 0 = 1$ for all s which are not in A. Thus $\chi_{A^c} = 1 \oplus \chi_A$.

We note that $\chi_A \cdot \chi_B$ is $1 \cdot 1 = 1$ at all s in $A \wedge B$. When s is not in $A \wedge B$, at least one of χ_A or χ_B is 0, so that $\chi_A \cdot \chi_B$ is 0 at such s. Thus $\chi_{A \wedge B} = \chi_A \cdot \chi_B$.

We note that $(A^c \wedge B^c)^c = A \vee B$. Hence,

$$\chi_{A\vee B} = 1 \oplus \chi_{A^c\wedge B^c} =$$
$$1 \oplus \chi_{A^c} \cdot \chi_{B^c} =$$
$$1 \oplus (1 \oplus \chi_A) \cdot (1 \oplus \chi_B) =$$
$$1 \oplus 1 \oplus \chi_A \oplus \chi_B \oplus \chi_A \cdot \chi_B =$$
$$\chi_A \oplus \chi_B \oplus \chi_A \cdot \chi_B$$

We note that $A \setminus B = A \wedge B^c$. Hence,

$$\chi_{A\setminus B} = \chi_{A\wedge B^c} =$$
$$\chi_A \cdot \chi_{B^c} = \chi_A \cdot (1 \oplus \chi_B) =$$
$$\chi_A \oplus \chi_A \cdot \chi_B$$

For the reverse identification we use $\chi_A \cdot \chi_A = \chi_A$ and $\chi_A \oplus \chi_A = 0$.

As we saw above Adding $\chi_A \cdot \chi_B$ to both sides

$$\chi_A \oplus \chi_B =$$
$$\chi_A \oplus \chi_B \oplus \chi_A \cdot \chi_B \oplus \chi_A \cdot \chi_B =$$
$$\chi_A \cdot (1 \oplus \chi_B) \oplus \chi_B(1 \oplus \chi_A) =$$
$$\chi_{A\wedge B^c} \oplus \chi_{A^c\wedge B}$$

Now,

$$\chi_{A\wedge B^c} \cdot \chi_{A^c\wedge B} = \chi_{A\wedge B^c\wedge A^c\wedge B} = 0$$

So

$$\chi_{(A\wedge B^c)\vee(A^c\wedge B)} = \chi_{A\wedge B^c} \oplus \chi_{A^c\wedge B} = \chi_A \oplus \chi_B$$

As seen above $1 \oplus \chi_A = \chi_{A^c}$.

Finally,

$$\chi_A \cdot (1 \oplus \chi_A) = \chi_A \oplus \chi_A \cdot \chi_A = \chi_A \oplus \chi_A = 0 = \chi_\phi$$

3. Assume that a class has students from each part of India (North/South/East/West), comprising of Boys and Girls, so that each of the letters of the English alphabet is the starting letter for exactly one student of each sex from each region. In other words, the information (Region, Sex, Starting Letter) uniquely specifies a student. Let A be the property that a student is from the North, B be the property that the student is a Boy, C be the property that the student's name starts the letter 'C' and D be the property that the student plays basketball for IISER Mohali. Explain the meaning of each set in the list below:

$$D^c; A \wedge B; A^c \wedge B^c \wedge D^c; D \setminus C$$

Solution:

1. The chosen student does not play basketball.
2. The chosen student is a Boy from the North.
3. The chosen student is not from the North, is not a Boy and does not play Basketball.
4. The chosen student plays Basketball and the name does not start with 'C'.

4. Suppose A, B and C are mutually exclusive events. Show (using only the laws of probability) that $P(A \vee B \vee C) = P(A) + P(B) + P(C)$.

Solution: First of all we note that for *any* events M and N,

$$P(M \vee N) = P(M) + P(N) - P(M \wedge N) \leq P(M) + P(N)$$

In particular, if $P(M) = 0$ and $P(N) = 0$ then $P(M \vee N) = 0$ *without* any further assumption on M and N.

We note that the mutually exclusive hypothesis means that

$$P(A \wedge B) = 0 = P(B \wedge C) = 0 = P(C \wedge A)$$

It follows from the laws of probability that

$$P(A \vee B) = P(A) + P(B)$$

Now, by the rules of Boolean algebra

$$(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C)$$

By the above argument we have

$$P(A \vee B) \wedge C) = P((A \wedge C) \vee (B \wedge C)) \leq P(A \wedge C) + P(B \wedge C) = 0$$

Hence, $A \vee B$ and C are *also* mutually exclusive. It follows that

$$P((A \vee B) \vee C) = P(A \vee B) + P(C) = P(A) + P(B) + P(C)$$

It is evident how we can extend this to any finite number of mutually exclusive events. (For infinite collections of sets we need to be a bit careful!)

5. Consider the set S of students in IISER Mohali. We pick a student at random (all students are equally likely to be chosen). Someone asserts that the probability that the student is from Hostel 8 is 0.3, and that the probability that it is a Male student is 0.6, and the probability that the student's name starts with 'Q' is 0.05. Finally, the person also says that the probability that it is not a Male student and not from Hostel 8 and that the name starts with 'A' is 0.7. Is it possible that all four estimates of probability are correct? If not, why not?

> **Solution:** Let A be the event that the student is from Hostel 8, let B be the event that the student is Male, and C be the event that the student's name starts with 'Q'. Finally, let D be the event that the students name starts with 'A'. Then $C \wedge D$ is empty or equivalently $D \subset C^c$.
>
> We are given $P(A) = 0.3$, $P(B) = 0.6$, $P(C) = 0.05$ and $p(A^c \wedge B^c \wedge D) = 0.7$. The laws of probability give us:
>
> $$\begin{aligned} P(A^c \wedge B^c \wedge D) &\leq P(B^c) \\ P(B^c) &= 1 - P(B) \end{aligned}$$
>
> But this gives the contradition that $0.7 \leq 0.4$ which is impossible.

6. The probability that a car jumps the red light at Tribune Chowk is 0.01. The probablity that a car in Chandigarh has a white color is 99.9%. Is it possible that these are mutually exclusive events? Should the police issue registration only to white cars? Discuss!

> **Solution:** If A is the event that a car jumps a red light, then $P(A) = 0.01$.
>
> If B is the event that a car is of white colour, then $P(B) = 99.9/100 = 0.999$.
>
> Since $P(A) + P(B) > 1$, the events are *not* exclusive. Since it is possible for owners of white cars to also jump the red light, police should not give registration only to white cars. This will not solve the problem!

7. If A and B are independent events, then show that A and B^c are independent events as well.

> **Solution:** We have $P(A \wedge B) = P(A)P(B)$. We also have, by the laws of probability
>
> $$P(A \wedge B) + P(A \wedge B^c) =$$
> $$P((A \wedge B) \vee (A \wedge B^c)) + P((A \wedge B) \wedge (A \wedge B^c)) =$$
> $$P(A) + P(\phi) = P(A)$$

It follows that

$$P(A \wedge B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

8. Four digits (0 to 9) are chosen at random (all digits are equally likely) in order to make a 4-digit number (which *may* start with 0). What is the probability that the number is divisible by 5? What is the probability that the number is divisible by 3?

   Now do the same problem *given* that the number does not start with 0.

Solution: Let $E(n)$ denote the event that the number obtained is n for n between 0000 and 9999. It is clear that $E(n)$ are mutually exclusive and exhaustive.

When we say a digit is chosen "at random" we are saying that each digit is equally likely. Hence, each $E(n)$ is equally likely and so $P(E(n)) = 1/10000$.

In the first case we are taking the A to be the union of events $E(n)$ where n is divisible by n. There are $10000/5 = 2000$ such numbers. So we see that the probability $P(A)$ in this case is $1/5 = 2000/10000$.

In the second case we are taking B to be the union of events $E(n)$ where n is divisible by 3. There are $9999/3 + 1 = 3334$ such cases. So the probability $P(B)$ in this case is $3334/10000$ which is slightly better than $1/3$.

Now, we are only taking numbers that don't start with 0. Thus we are looking at $C = \vee_n E(n)$ only for $n \geq 1000$ and we want to calculate the conditional probability $P(A|C)$ and $P(B|C)$. To do so we need to calculate $P(C)$, $P(A \wedge C)$ and $P(B \wedge C)$.

We see that $P(C) = 9000/10000 = 9/10$ since C is the union of $E(n)$ for 9000 values of n.

We note that $A \wedge C$ is the union of $E(n)$ where $n \geq 1000$ and n divisible by 5. We calculate this to be $9000/5 = 180$. The probability $P(A|C) = 180/9000 = 1/5$. Note that this says that A is *independent* of C.

We note that $B \wedge C$ is the union of $E(n)$ where $n \geq 1000$ and n divisible by 3. We calculate this to be $9000/3 = 3000$. The probability $P(B|C) = 3000/9000 = 1/3$. Note that this says that B is *not* independent of C.

This difference between the divisibility by 3 and 5 is worth trying to understand under the condition that n is at least 1000 is worth trying to understand!

9. (Starred) Three of us go out for ice-cream, but we have money only for two ice-creams. We don't want to share, so one of us has to do without ice-cream, but we want to do this fairly. Unfortunately, all we have is a coin which is fair. How do we we set up a game with a sequence of coin flips so that the probability that any one of us wins it is 1/3?