Review of Analysis

Since we are going to use limits in a significant way, it will be good to review some key ideas from analysis.

Asymptotic Behaviour

It is useful to compare the behaviour of two functions $f(x)$ and $g(x)$ as x approaches some value (say 0) or approaches infinity. Since x approaching x_0 is the same as $1/(x - x_0)$ approaching infinity and that is the most interesting case for us, we will state the conditions only in the case of x approaching infinity.

We say that $f(x) = O(g(x))$ as x approaches infinity if there is a constant M so that $f(x) \leq Mg(x)$ for all sufficiently large x (more precisely, for $x > N$ for some N). In other words, $f(x)$ is bounded by a constant multiple of $g(x)$ when x is sufficiently large.

We say that $f(x) = o(g(x))$ as x approaches infinity if for any positive integer n, $f(x) < g(x)/n$ for sufficently large x (more precisely, for $x > N_n$ for some N_n, which is allowed to depend on n). In other words, $f(x)$ is much smaller than $g(x)$ for sufficiently large x.

We say that $f(x) \simeq g(x)$ as x approaches infinity if $f(x)/g(x)$ approaches 1 as x approaches infinity. In this case we say that $f(x)$ and $g(x)$ are asymtotically equal as x approaches infinity.

A Poynomial $P(x) = c_0 x^k + c_1 x^{k-1} + \ldots_c k$ with $c_0 \neq 0$ is called a polynomial of degree k. We have $P(x) = O(x^k)$ and $P(x) = o(x^m)$ for $m > k$. In fact, we have $P(x) \simeq c_0 x^k$.

Some Series and Functions

We have already discussed the geometric series $\sum_{k=0}^{\infty} x^k$ which converges for $|x| < 1$ to $1/(1-x)$. We replace x by $-x$ to get $\sum_{k=0}^{\infty}(-x)^k$ as summing to $1/(1+x)$ for $|x| < 1$.

In fact, this convergence is *absolute* for $|x| < 1 - 1/r$ for any positive r. This makes the formulas work even if we integrate or differentiate *term by term*! So, by differentiating the first series r times and dividing by $r!$, we get the formula:

$$\frac{1}{(1-x)^{r+1}} = \sum_{k=0}^{\infty} \binom{k+r-1}{r} x^k$$

We have already used this formula in order to study the Negative Binomial distribution.

Integrating this series once, we obtain:

$$\int_0^x \frac{dt}{1+t} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$$

The series (on the right-hand side) converges only for $|x| < 1$. However, the integral (on the left-hand side) makes sense for all values of x. Hence, we define $\log(x) = \int_1^x (dt/t)$ and only use the series expansion for $|x| < 1$.

The integral $\log(x) = \int_1^x (dt/t)$ has the interesting property

$$\log(xy) - \log(x) = \int_x^{xy} (dt/t) = \int_1^y (du/u) = \log(y)$$

obtained by the substitution $u = xt$. In other words, $x \mapsto \log(x)$ is a group isomomorphism from the *multiplicative* group of positive real numbers to the *additive* group of all real numbers. This makes it a very useful function. (Note that for Mathematicians log *always* means the *natural* logarithm.) It follows that for each positive x there is a unique y so that $\log(y) = x$. Moreover, log is order-preserving.

The inverse homomorphism exp is defined by $\log(\exp(x)) = x$. By the chain rule, we see that

$$1 = \frac{d}{dx}\log(\exp(x))) = \frac{d\log(t)}{dt}|_{t=\exp(x)}\frac{d\exp(x)}{dx} = \frac{1}{\exp(x)}\frac{d\exp(x)}{dx}$$

It follows that $d\exp(x)/dx = \exp(x)$. Thus, the Taylor series for $\exp(x) = \sum_{k=0}^{\infty} x^k/k!$. By the comparison test one can show that this series converges for all values of x.

Since exp is the inverse of a group isomorphism, it too is a group isomorphism from the additive group of real numbers to the multiplicative group of positive real numbers. For this reason, we often use the notation $e = \exp(1)$ and write $e^x = \exp(x)$.

Asymptotic formulas

The function $1/t$ can be compared with the step functions $l(t) = 1/\lfloor t \rfloor$ and $c(t) = 1/\lceil t \rceil$; we have $c(t) \le 1/t \le l(t)$ for $t \ge 1$. It follows that

$$\sum_{k=2}^{n}(1/k) \le \log(n) \le \sum_{k=1}^{n-1}(1/k)$$

We see that $\sum_{k=1}^{n}(1/k)$ is asymptotic to $\log(n)$. In fact, we see that

$$\sum_{k=1}^{n}\frac{1}{k} - \log(n) \le 1$$

is a bounded increasing sequence and so it converges to a constant γ; this constant is called the Euler-Mascheroni constant. In particular, since $\log(n)$ goes to infinity as n goes to infinity, we have:

$$\sum_{k=1}^{n} 1/k \to \infty \text{ as } n \to \infty$$

We have $d\log(x)/dx = 1/x < 1$ for $x > 1$. This means that $\log(x) < (1-x)$ for $x > 1$ by integrating both sides. On the other hand $d\log(x)/dx = 1/x > 1$ for $0 < x < 1$. This again means that $\log(x) < x - 1$ for $0 < x < 1$ by integrating both sides. Since $\log(1) = 0$, we see that

$$\log(x) < x - 1 \text{ for all } x > 0$$

When x is large, we can do better. We have $d\log(x)/dx = 1/x < 1/n$ for $x > n$. It follows that $\log(x) < \log(n) + (1/n)x$ for $x > n$. From this we can deduce that $\log(x) = o(x)$ as x approaches infinity.

The initial expansion of $\log(1+x) = x - x^2/2 + g(x)$ gives us

$$|g(x)| \leq \sum_{k=3}^{\infty} x^k/k \leq (|x^3|/3) \sum_{k=0}^{\infty} |x|^k = \frac{|x|^3}{3(1-|x|)}$$

for $|x| \leq 2/3$ we see that this is $\leq |x|^3/3$.

Fixing x and applying this to x/n for large n, we see see that $n\log(1+x/n) \simeq x$ as n approaches infinity. It follows that $\exp(n\log(1+x/n)) \simeq \exp(x)$ as n approaches infinity. This is usually written as

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Moreover, from the fact that $\log(1+x/n) < x/n$ proved above, we see that this limit is an increasing one.

We easily check that $d(x\log(x) - x)/dx = \log(x)$. It follows that

$$\int_1^n \log(t)dt = n\log(n) - n + 1$$

We can form the Trapezoidal approximation of the integral on the left-hand side

$$\sum_{k=1}^{n-1} \frac{\log(k+1) - \log(k)}{2} = \sum_{k=1}^{n} \log(k) - \frac{1}{2}\log(n) = \log(n!) - \frac{1}{2}\log(n)$$

The *error* in this approximation is given by

$$d_n = \log(n!) - (n + 1/2)\log(n) + n - 1$$

We will not go through a computation to show that this error approaches a fixed limit as n goes to infinity. We compute

$$d_{n+1} - d_n = (n + 1/2)\log(1 + 1/n) + 1$$

Using the expression $\log(1 + x) = x - x^2/2 + g(x)$ we get

$$d_{n+1}-d_n = (n+1/2)\left(\frac{1}{n} - \frac{1}{2n^2} + g(1/n)\right)-1 = \left(1 - \frac{1}{4n^2}\right)-1+(n+1/2)g(1/n)$$

For $n > 2$ we have $|g(1/n)| < 1/3n^3$ so that

$$|d_{n+1} - d_n| \leq \frac{1}{4n^2} + \frac{n + 1/2}{3n^3} \leq c/n^2$$

for some suitable constant c. Since $\sum_{n=1}^{\infty} c/n^2$ converges, by the comparison test for series, we see that $\sum_{n=1}^{\infty}(d_{n+1} - d_n)$ converges to some number. The sum $\sum_{n=1}^{N-1}(d_{n+1} - d_n)$ is easily calculated to be $d_N - d_1$. Hence, we see that d_N converges to some constant C as N approaches infinity. In other words,

$$\lim_{n\to\infty} (\log(n!) - (n + 1/2)\log(n) + n - 1) = C$$

Exponentiating both sides gives us

$$\frac{n!}{n^n e^{-n}\sqrt{n}} \to K \text{ as } n \to \infty$$

for a suitable constant K. (We will eventually determine K as well!) The approximation

$$n! \simeq K n^n e^{-n}\sqrt{n} \text{ for large } n$$

was found by de Moivre; after putting in the value of K it is called Stirling's approximation.