Solutions to Assignment 2

1. A four-sided dice (with faces showing different numbers) is rolled three times. What is the probability that the numbers obtained are different?

> **Solution:** The atomic events are of the form $F(a, b, c)$ where a, b and c are the faces that appear. As we run over all distinct triples (a, b, c) we obtain events that are mutually exclusive and exhaustive. There are $4^3 = 64$ such events.
>
> We *assume* that the dice is fair and so each $F(a, b, c)$ has probability $1/64$.
>
> The event D we are looking for is the $\vee$ of all $F(a, b, c)$ where $a \neq b \neq c \neq a$. We put
>
> $$T = \{(a, b, c) | a \neq b \neq c \neq a\}$$
>
> There are 4 choices for a, after which there are only 3 choices for b and then there are only 2 choices left for c. It follows that T has $24 = 4 \cdot 3 \cdot 2$ elements.
>
> The event we want is $D = \vee_{(a,b,c)\in T} F(a, b, c)$. Since these events are mutually exclusive and each has probability $1/64$ we see that $P(D) = 24/64 = 3/8$.

2. Player A flips a coin 12 times with the hope of getting at least 2 heads. Player B rolls dice 6 times with the hope of getting at least 1 six. Which player has a better probability of success?

> **Solution:** As usual, we *assume* that the coin and dice are fair!
>
> For a fair coin, the probability of the event S_r corresponding to r heads in 12 flips is given by the binomial distribution as
>
> $$P(S_r) = \binom{12}{r} \frac{1}{2^{12}}$$
>
> The events S_r are mutually exclusive. If A denotes the event of success for A, then A is the union of S_r for $r \geq 2$. In other words, we have
>
> $$P(A) = \sum_{r=2}^{12} \binom{12}{r} \frac{1}{2^{12}} = 1 - \frac{13}{2^{12}}$$
>
> For a fair dice, the probability of the T_s corresponding to s sixes in 6 tosses is given by the binomial distribution as
>
> $$P(T_s) = \binom{6}{s} \frac{5^{6-s}}{6^6}$$

The events T_s are mutually exclusive. If B denotes the event of success for B, then B is the union of T_s for $s \geq 1$. In other words, we have

$$P(B) = \sum_{r=1}^{6} \binom{6}{r} \frac{5^{6-s}}{6^6} = 1 - \frac{5^6}{6^6}$$

We note that $5^6 \cdot 2^{12} = 64 \cdot 10^6$ while

$$6^6 \cdot 13 < (2.2)^2 \cdot 10^4 \cdot 13 < 65 \cdot 10^4$$

So $P(B) < P(A)$ by a lot. So A easily wins!

3. Suppose we have a collection of $100n$ samples, which are uniformly distribution among 100 values so that each value is taken *exactly* n times. Calculate the mean, median, mode and variance.

Solution: We note that the mean is

$$\frac{\sum_{k=1}^{100} k \cdot n}{100 \cdot n} = \frac{100 \cdot (100+1)}{2 \cdot 100} = 50.5$$

(In the first numerator k is the value and n is the number of times the value is taken.)

To get the median we need to find the smallest m such that

$$50 \cdot n < \sum_{k=1}^{m} n = m \cdot n$$

So $m = 51$.

Since all values are taken with equal frequency, there is *no* well-defined mode!

We note that the variance is

$$\begin{aligned} \frac{\sum_{k=1}^{100} (k-50.5)^2 \cdot n}{100 \cdot n} &= \frac{\sum_{k=1}^{100} (k^2 - 101 \cdot k + 2550.25)}{100} \\ &= \frac{100 \cdot (100+1) \cdot (2 \cdot 100 + 1)}{6 \cdot 100} - 101 \frac{100 \cdot (100+1)}{2 \cdot 100} + 2550.25 \\ &= 3383.5 - 10201/2 + 2550.25 = 833.25 \end{aligned}$$

4. A standard (6-sided) dice is rolled two times. Let X_i denote the value obtained on the i-th roll. Is this a random variable? What about $S = X_1 - X_2$, the difference of the values rolled? What are the distributions of each of these random variables?

Solution: Yes, each X_i is a random variable and $S = X_1 - X_2$ is also a random variable. The distribution of each X_i is given by $P(X_i = j) = 1/6$ for $j = 1, \dots, 6$. The distribution for S is a bit more complicated to calculate. The events $X_1 = j$ and $X_2 = k$ are independent so:

$$\begin{aligned}
P(S=-5) &= P(X_1 = 1 \wedge X_2 = 6) && = 1/36 \\
P(S=-4) &= P((X_1 = 1 \wedge X_2 = 5) \vee (X_1 = 2 \wedge X_2 = 6)) && = 2/36 \\
P(S=-3) &= P(\vee_{k=1}^{3}(X_1 = k \wedge X_2 = 3+k)) && = 3/36 \\
P(S=-2) &= P(\vee_{k=1}^{4}(X_1 = k \wedge X_2 = 2+k)) && = 4/36 \\
P(S=-1) &= P(\vee_{k=1}^{5}(X_1 = k \wedge X_2 = 1+k)) && = 5/36 \\
P(S=0) &= P(\vee_{k=1}^{6}(X_1 = k \wedge X_2 = k)) && = 6/36
\end{aligned}$$

By symmetry we see that $P(S = k) = P(S = -k)$ since the values on first and second dice have to be interchanged in this case.

5. A player has a fair coin, and a standard 4-sided dice and a standard 6-sided dice. The player flips the coin, if it shows head, then the 4-sided dice is rolled and its value noted; else if the coin shows tail then the 6-sided dice is rolled and it value noted. Is the result X, a random variable? What are the values of $P(X = i)$?

Solution: Yes, X is a random variable. Let H denote the event that the coin shows head. Let F be the random variable that is the value on the 4-sided dice and S the the random varialbe that is the value on the 6-sided dice. All three are independent so:

$$\begin{aligned}
P(X=1) &= P((H \wedge F = 1) \vee (H^c \wedge S = 1)) && = (1/2)(1/4 + 1/6) = 5/24 \\
P(X=6) &= P(T \wedge S = 6) && = (1/2)(1/6) = 1/12
\end{aligned}$$

The cases $P(X = i)$ for $i = 2, 3, 4$ are similar to $P(X = 1)$ and the case $P(X = 5)$ is similar to $P(X = 6)$.

6. A die is rolled repeated until we get a 6. The number of rolls is recorded. Let X denote the random variable that denotes the number of rolls.

   1. Calculate the value of X for which the probability is the highest.
   2. What is the smallest s so that $P(X \leq s) \geq 1/2$?

Solution: We have seen that $P(X = n) = (5/6)^{n-1}(1/6)$. Clearly this is largest for $n = 1$!

To get the median, we need to use the formula $\sum_{n=1}^{N} t(1-t)^{n-1} = 1-(1-t)^N$. This gives us

$$\sum_{n=1}^{N}(1/6)(5/6)^{n-1} = 1 - (5/6)^N$$

In order for this to be at least half, we need $(5/6)^N \leq 1/2$. Taking log of both sides gives $N \log(5/6) \leq -\log(2)$. Thus, we need $N \geq \log(2)/\log(6/5) = 3.80$ or $N \geq 4$.

7. A fair dice is rolled until we see a 6. What is the probability that we never see a 6?

Solution: Let E_n be the event that we do not see a 6 in n rolls of the dice. Then $E_{n+1} \subset E_n$ is a *decreasing* chain of events. The event E, that we do not see a 6 at all, is $\wedge_n E_n$. By the law of infinity in probability we see that $P(E) = \inf_n P(E_n)$. Now $P(E_n) = (5/6)^n$. Since $0 < 5/6 < 1$, we see that $\inf_n (5/6)^n = 0$.

Thus the probability that we do not see a 6 at all is 0!

8. A fair dice is rolled until we see a number different from 6. Let F be the random variable that is the number that we see. What is the probability distribution of F?

Solution: Let W be the random variable that designates the number of throws needed. Then the event $(W, F) = (n, i)$ is the event that we threw $n-1$ 6's followed by a throw of i (which is different from 6). Thus,

$$P((W, F) = (n, i)) = (1/6)^{n-1}(1/6) = (1/6)^n$$

Now, the events $W = n$ for different n are exclusive, thus $F = i$ is the *exclusive* union of $(W, F) = (n, i)$ for different n.

$$P(F = i) = \sum_{n=1}^{\infty} P((W, F) = (n, i)) = \sum_{n=1}^{\infty}(1/6)^n = \frac{1/6}{1-1/6} = 1/5$$

We can also argue by symmetry that each $F = i$ is equally likely, these are mutually exclusive and exhaustive, hence each has probability 1/5!

9. A fair dice is rolled and a fair coin is flipped. If we see a tail, then we take the value of the dice; if we see a head then we add 6 to the value of the dice *unless* we see a 6; if we

see a head and a 6 then we repeat the experiment. What is the probability distribution of the random variable X which we obtain as the value?

Solution: This is just a more complicated version of the previous problem!

Let W count the number of experiments needed to be performed. The event $(W, X) = (n, i)$ means that $n-1$ times we got a head *and* a 6. Finally, we got the value i which is from 1 to 11; either getting a tail and the value i on the dice or by getting a head and the value $i-6$ on the die. It follows that

$$P((W, X) = (n, i)) = (1/12)^{n-1}(1/12) = (1/12)^n$$

As above, we see that $W = n$ are mutually exclusive for different n and exhaustive, thus

$$P(X = i) = \sum_{n=1}^{\infty} P((W, X) = (n, i)) = \sum_{n=1}^{\infty} (1/12)^n = \frac{1/12}{1 - 1/12} = 1/11$$

We can also argue by symmetry as above.

10. In a "random" chemistry experiment (we do not have precise control over the outcome) we can observe whether the solution is acidic (event A) and whether the solution is coloured (event B). Assume that $P(B) > 0$. We now carry out the experiment repeatedly until B is observed. Express the probability that A is observed at the same time as B in terms of the probabilities of events A and B and their unions and intersections.

Solution: This is an abstract version of the above. The repeatable experiments are assumed to be independent. Let W be the number of experiments until we observe event B. Then $P(W = n) = (1 - P(B))^{n-1}P(B)$ as seen earlier. The probability that we also observe A at the n-th stage is $(1 - P(B)^{n-1}P(A \wedge B)$. Thus the probability, that we observe A when we observe B is

$$\sum_{n=1}^{\infty} (1 - P(B))^{n-1} P(A \wedge B) = \frac{P(A \wedge B)}{1 - (1 - P(B))} = \frac{P(A \wedge B)}{P(B)} = P(A|B)$$

Hence, this gives another way of interpreting $P(A|B)$: we perform the experiment repeatedly until we observe B, then $P(A|B)$ is the probability that we simultaneously observe A.

11. We know that 60% of the dogs in the campus are black. We repeatedly observe dogs until we find one that is black; in that case we record its age before releasing it. After 20 such recordings we find that 5 of these black dogs were puppies. Assuming that frequency

is a good estimate of probability, what is a reasonable estimate of the percentage of black puppies in the campus dog population? Can we conclude the 25% of the dogs are puppies?

Solution: We note that there have been 20 *recordings*. Since we only record the results for *black* dogs, this means that thare are 20 recordings of black dogs out of which 5 are puppies.

If B denotes the event of the dog being black and P is the event of the dog being a puppy, then we have recording the frequence of $(P|B)$ as $5/20 = 0.25$. Since frequency is meant to be a good estimate of probability $P(P|B) = 0.25$.

Since we are given the $P(B) = 0.6$ (or 60%), it follows that $P(P \wedge B) = (0.25) \cdot (0.6) = 0.15$. In other words, we estimate that 15% of the dogs are black puppies.

We *cannot* conclude that 25% of the dogs are puppies. We must *also* do a similar experiment where we only record the species of non-white dogs.

Suppose that we do the same experiment only recording the species of non-white dogs and in this case 10 out of 20 non-white dogs turn out to be puppies. This means that $P(P|B^c) = 0.5$. It follows that $P(P \wedge B^c) = (0.5)(1 - 0.6) = 0.3$ Putting these together we would get $P(P) = P(P|B) + P(P|B^c) = 0.45$ In other words, the percentage of puppies is estimated as 45% in the campus population.