Solutions to Assignment 13

1. Given a complex inner product space V and a linear transformation $A : V \to V$. Suppose B is a *set-theoretic* map such that $\langle Av, w\rangle = \langle v, Bw\rangle$ for all v, w in V. Is it true that B is a linear map?

Solution: We check that for all v, w in V and for all complex numbers a, we have

$$\langle v, Baw\rangle = \langle Av, aw\rangle = \overline{a}\langle Av, w\rangle = \overline{a}\langle v, Bw\rangle = \langle v, aBw\rangle$$

Since this is true for *all* v in V. It follows that $Baw = aBw$. (For example, we could take $v = Baw - aBw$ and then we would get $\langle v, v\rangle = 0$.) Similarly, for all u, v and w in V, we have

$$\langle v, B(u + w)\rangle = \langle Av, u + w\rangle = \langle Av, u\rangle + \langle Av, w\rangle = \langle v, Bu\rangle + \langle v, Bw\rangle = \langle v, Bu + Bw\rangle$$

As above, this means that $B(u + w) = Bu + Bw$. It follows that B is linear.

2. Consider the linear transformation $A : \mathbb{C}^2 \to \mathbb{C}^2$ given by interchanging the two coordinates. Is it unitary? Is it self-adjoint? What is the matrix M of this linear transformation in the bases $e_1 = (1, 0)$, $e_2 = (1, 1)$. Is M a unitary or Hermitian matrix? Why not?

Solution: We have

$$\langle A(z_1, z_2), (w_1, w_2)\rangle = z_2\overline{w_1} + z_1\overline{w_2} = \langle (z_1, z_2), A(w_1, w_2)\rangle$$

Similarly,

$$\langle A(z_1, z_2), A(w_1, w_2)\rangle = z_2\overline{w_2} + z_1\overline{w_1} = \langle (z_1, z_2), (w_1, w_2)\rangle$$

So A is unitary and self-adjoint. We also note that the matrix A is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

This is neither Hermitian or unitary. This is because the basis $\{e_1, e_2\}$ is not orthogonal.

3. Write down the linear transformation that is an orthogonal projection to the subspace generated by $(1,-1,0)$ and $(1,0,-1)$.

Solution: The direct approach is to form the orthogonal basis out of the given vectors $v_1 = (1,-1,0)$ and $v_2 = (1,0,-1)$. We put $w_1 = v_1$ and

$$w_2 = v_2 - \frac{1}{2}v_1 = \frac{1}{2}(1,1,-2)$$

The projection map is then given by

$$Pw = \frac{\langle w, w_1\rangle}{2}w_1 + 2\frac{\langle w, w_2\rangle}{3}w_2$$

where we have used $\langle w_1, w_1\rangle = 2$ and $\langle w_2, w_2\rangle = 3/2$. We note that, in this particular case the vector $w_3 = (1,1,1)$ is perpendicular to v_1 and v_2, so the projection could also be given by

$$Pw = w - \frac{\langle w, w_3\rangle}{3}w_3$$

where we have used $\langle w_3, w_3\rangle = 3$.

4. Classify the following matrices as unitary, Hermitian, idempotent, orthogonal idempotent, normal or none of these.

(a)

$$\begin{pmatrix} 0 & \iota \\ -\iota & 0 \end{pmatrix}$$

Solution: Hermitian and unitary. An invertible idempotent is identity which it is not. A unitary or a Hermitian matrix is normal, so it is normal.

(b)

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Solution: Hermitian and idempotent, hence it is orthogonal idempotent. Since rank is 1, it is not unitary. Since it is Hermitian, it is normal.

(c)

$$\begin{pmatrix} 1 & \iota \\ -\iota & 1 \end{pmatrix}$$

Solution: It is Hermitian. It is not unitary since norm of 1st column is 2. It is of rank 2 and not identity so it is not idempotent. It is normal since it is Hermitian.

(d)

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Solution: It is unitary but not hermitian. It is of rank 2 and not identity, hence it is not idempotent. It is normal since it is unitary.

(e)

$$\begin{pmatrix} \iota\cos t & \iota\sin t \\ -\iota\sin t & \iota\cos t \end{pmatrix}$$

Solution: Since it is ι multiple of unitary, it is unitary. Since diagonal entries are not real, it is not Hermitian. Since it is of rank 2 and not identity it is not idempotent. It is normal since it is unitary.

(f)

$$\begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix}$$

Solution: It is unitary since diagonal entries are of absolute value 1. Since diagonal entries are not real, it is not Hermitian. Since it is of rank 2 and not identity it is not idempotent. It is normal since it is unitary.

5. Consider the 2×2 invertible matrix

$$G = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

Write the KAN, KP and KAK decompositions for this matrix. Are the resulting matrices real or complex?

Solution: We calculate

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \frac{6+15}{4+9}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3/13 \\ 2/13 \end{pmatrix}$$

so

$$G \cdot \begin{pmatrix} 1 & -21/13 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3/13 \\ 3 & 2/13 \end{pmatrix}$$

We note that $\|(2,3)\|^2 = 13$ so

$$G \cdot \begin{pmatrix} 1 & -21/13 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{13} & 0 \\ 0 & \sqrt{13} \end{pmatrix} = \begin{pmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{pmatrix}$$

The latter is an orthogonal matrix. Thus we have the KAN decomposition

$$G = \begin{pmatrix} 2/\sqrt{13} & -3/\sqrt{13} \\ 3/\sqrt{13} & 2/\sqrt{13} \end{pmatrix} \begin{pmatrix} \sqrt{13} & 0 \\ 0 & 1/\sqrt{13} \end{pmatrix} \begin{pmatrix} 1 & 21/13 \\ 0 & 1 \end{pmatrix}$$

Next, we note that G is itself symmetric and positive definite ($2 > 0$ and determinant positive). Hence, G is the square-root of $G^2 = G^t G$. In other words, the KP decomposition of G is $1 \cdot G$!

To find the KAK decomposition, we look for an orthonormal basis of eigenvectors for G. The characteristic polynomial is $T^2 - 7T + 1$, which has roots $(7 \pm 3\sqrt{5})/2$. Putting $\phi = (1 + \sqrt{5})/2$, this is $2 + 3\phi$ and $5 - 3\phi$ (note that both the numbers are positive). We see that

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ \phi \end{pmatrix} = \begin{pmatrix} 2 + 3\phi \\ 3 + 5\phi \end{pmatrix}$$

Note that $\phi^2 = 1 + \phi$, so

$$\phi(2 + 3\phi) = 2\phi + 3(1 + \phi) = 3 + 5\phi$$

So $(1, \phi)$ is an eigenvector. Since the matrix is symmetric $(-\phi, 1)$ is another eigenvector. It follows that

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -\phi \\ \phi & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\phi \\ \phi & 1 \end{pmatrix} \begin{pmatrix} 2 + 3\phi & 0 \\ 0 & 5 - 3\phi \end{pmatrix}$$

We then get

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = \frac{1}{\sqrt{2 + \phi}} \begin{pmatrix} 1 & -\phi \\ \phi & 1 \end{pmatrix} \begin{pmatrix} 2 + 3\phi & 0 \\ 0 & 5 - 3\phi \end{pmatrix} \frac{1}{\sqrt{2 + \phi}} \begin{pmatrix} 1 & \phi \\ -\phi & 1 \end{pmatrix}$$

As the KAK decomposition

6. (Starred) Will it always be true that for real matrices the KAN, KP and KAK decompositions will give real matrices?

Solution: Given that G is a real matrix, G^tG is also a real positive definite symmetric matrix. Hence, it can be diagonalised by an real orthogonal matrix V, so that $VG^tGV^t = B$ is a diagonal matrix with positive entries on the diagonal. We then take A to be the natural diagonal sqaure root of B with positive diagonal entries; it is also real. Hence, $U = GV^tA^{-1}$ is also a real matrix and we check that it is orthogonal. Thus, the KAK decomposition of G is UAV which consists of real matrices. We note that $K = UV$ is orthogonal and $P = V^tAV$ is positive definite; both these matrices are real. Moreover, $G = KP$ is the KP decomposition of G.

Note: In the process of the above argument we showed that if UAV is a KAK decomposition of a matrix G, then $(UV)(V^tAV)$ is the KP decomposition of the same matrix G!

The Gram-Schmidt process for G does not involve an operations with complex numbers, hence the KAN decomposition also consists of real matrices.

7. Check that the following matrix is normal.

$$G = \begin{pmatrix} 1 & -\iota \\ -\iota & 1 \end{pmatrix}$$

Find the spectral decomposition of this matrix. (Equivalently, find an orthonormal basis of eigenvectors.)

Solution: We note that $(1, -\iota)$ and $(-\iota, 1)$ are unitarily orthogonal to each other since $-\bar{\iota} - \iota = 0$! Moreover, the norm of each vector is $\sqrt{2}$. Thus

$$G = \begin{pmatrix} 1/\sqrt{2} & -\iota/\sqrt{2} \\ -\iota/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

is the KP decomposition of the matrix. We note that the second matrix is a scalar matrix and hence every vector is an eigen vector for it!

It is clear that $(1, 1)$ is an eigen vector for G and so is $(1, -1)$ with eigen values $1 + i$ and $1 - i$ respectively. Hence, we have

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} G \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} (1 - i) & 0 \\ 0 & (1 + i) \end{pmatrix}$$

Here the matrix

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

is a orthogonal (and hence unitary) matrix. Thus we have found an orthogonal basis of eigen vectors. This also shows that G is normal.

We now do the proof in the more mundane algorithmic way!

We note that

$$G^* = \overline{G}^t = \begin{pmatrix} 1 & \iota \\ \iota & 1 \end{pmatrix}$$

Hence,

$$G^*G = \begin{pmatrix} 1 & \iota \\ \iota & 1 \end{pmatrix} \begin{pmatrix} 1 & -\iota \\ -\iota & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Similarly,

$$GG^* = \begin{pmatrix} 1 & \iota \\ \iota & 1 \end{pmatrix} \begin{pmatrix} 1 & -\iota \\ -\iota & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Now, the P component of the KP decomposition is $\sqrt{G^*G}$ which, in this case, is

$$P = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

Hence, the K component is

$$K = GP^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \iota \\ \iota & 1 \end{pmatrix}$$

This gives the KP decomposition and we then find the orthogonal basis of eigenvectors as before.