Solutions to Quiz 6

1. For the quaternion $q = 1 + 2\hat{\imath} - 2\hat{\jmath}$ carry out the following:

   (a) Calculate the norm of q.

   Solution: The norm of q is $1 + 2^2 + 2^2 = 9$.

   (b) Calculate the 3×3 matrix B of the linear transformation $(0, w) \mapsto q \odot (0, w) \odot q^{-1}$.

   Solution: We have

$$\begin{aligned}
\hat{\imath} &\mapsto (1/9)(1 + 2\hat{\imath} - 2\hat{\jmath})\hat{\imath}(1 - 2\hat{\imath} + 2\hat{\jmath}) \\
\hat{\jmath} &\mapsto (1/9)(1 + 2\hat{\imath} - 2\hat{\jmath})\hat{\jmath}(1 - 2\hat{\imath} + 2\hat{\jmath}) \\
\hat{k} &\mapsto (1/9)(1 + 2\hat{\imath} - 2\hat{\jmath})\hat{k}(1 - 2\hat{\imath} + 2\hat{\jmath})
\end{aligned}$$

   This gives

$$\begin{aligned}
\hat{\imath} &\mapsto (1/9)(\hat{\imath} - 8\hat{\jmath} + 4\hat{k}) \\
\hat{\jmath} &\mapsto (1/9)(\hat{\jmath} - 8\hat{\imath} + 4\hat{k}) \\
\hat{k} &\mapsto (1/9)(-4\hat{\jmath} - 4\hat{\imath} - 7\hat{k})
\end{aligned}$$

   Thus the matrix is

$$B = \begin{pmatrix} 1/9 & -8/9 & -4/9 \\ -8/9 & 1/9 & -4/9 \\ 4/9 & 4/9 & -7/9 \end{pmatrix}$$

   (c) Is B an orthogonal matrix?

   Solution: We know that for any non-zero quaternion $O(q) : (0, w) \mapsto q \odot (0, w) \odot q^{-1}$ is an orthogonal matrix. In particular, B is an orthogonal matrix.

   (d) What is a vector fixed by B?

   Solution: We see that $(1, -1, 0)$ is fixed by B.