Solutions to Assignment 12

1. Write the norms, conjugates and inverses of the following complex numbers and quaternions.

   (a) $5 + 12\iota$

   Solution: The conjugate of $5 + 12\iota$ is $5 - 12\iota$ and the norm is

   $$(5 + 12\iota)(5 - 12\iota) = 5^2 + 12^2 = 13^2$$

   It follows that $(5 - 12\iota)/13^2$ is the inverse of $5 + 12\iota$.

   (b) $\sin t + \cos t\iota$.

   Solution: The conjugate of $\sin t + \cos t\iota$ is $\sin t - \cos t\iota$, hence its norm is

   $$(\sin t + \cos t\iota)(\sin t - \cos t\iota) = \sin^2 t + \cos^2 t = 1$$

   Hence, its conjugate is also its inverse.

   (c) $\cos s \cos t + \sin s \sin t\hat{i} + \sin s \cos t\hat{k} + \cos s \sin t\hat{k}$

   Solution: The conjugate of the given quaternion is $\cos s \cos t - \sin s \sin t\hat{i} - \sin s \cos t\hat{k} - \cos s \sin t\hat{k}$. Hence its norm is given by

   $$\cos^2 s \cos^2 t + \sin^2 s \sin^2 t + \sin^2 s \cos^2 t + \cos^2 s \sin^2 t = 1$$

   Hence, its conjugate is also its inverse.

   (d) $1 + \hat{i} + \hat{j} + \hat{k}$.

   Solution: The conjugate of the given quaternion is $1 - \hat{i} - \hat{j} - \hat{k}$. It follows that its norm is $1 + 1 + 1 + 1 = 4$. Hence its inverse is $(1 - \hat{i} - \hat{j} - \hat{k})/4$.

2. What are all the quaternions q so that $\hat{i}q\hat{i} = -q$.

   Solution: We note that $\hat{i}1\hat{i} = -1$ and $\hat{i}\hat{i}\hat{i} = -\hat{i}$ since $\hat{i}^2 = -1$. On the other hand $\hat{i}\hat{j} = -\hat{j}\hat{i}$ so that $\hat{i}\hat{j}\hat{i} = \hat{j}$ and similarly for $\hat{k}$. It follows that a quaternion satisfies the above property if and only if q is of the form $a + b\hat{i}$.

3. Given a quaternion (a, v) characterise all quaternions q so that $q \odot (a, v) = (a, v) \odot q$.

Solution: We note that $(a, v) = (a, 0) + (0, v)$ and $(a, 0)$ commutes with all quaternions. Thus it is enough to find out what quaternions satisfy $q \odot (0, v) = (0, v) \odot q$. We may assume that $v \neq 0$ and hence, by scaling v, we can assume that v is a unit vector. Since the quaternion operations are invariant under an orthogonal change of bases, we can assume that v is the unit vector $\hat{\imath}$. As seen above, a quaternion commutes with $\hat{\imath}$ if and only if it is of the form $b + c\hat{\imath}$. It follows that if $v \neq 0$ then the only quaternions that commute with (a, v) are those of the form (b, cv) for real numbers b and c.

4. For a fixed unit vector v define the map $w \mapsto v \times (w \times v)$ from 3-space to itself. Describe this map in words. Show that your description is correct by calculation.

Solution: We note when w is not linearly dependent on v, the vector $w \times v$ is perpendicular to w and v. Hence the vector $v \times (w \times v)$ is along the component $w_1 = w - \frac{w^t v}{v^t v} v$ of w which is perpendicular to v. Hence, we obtain a map $v \mapsto v \times (w \times v) = q(v) w_1$ for some function q on 3-space. Moreover, we easily see that q is a quadratic form. For v such that $v^t v = 1$ we check easily that $q(v) = 1$. It follows that $q(v) = v^t v$. Thus, we see that, for w linearly independent of v, we have

$$v \times (w \times v) = (v^t v) \left(w - \frac{w^t v}{v^t v} v \right)$$

We see that this is true for $v = w$ in which case both sides are 0. Hence,

$$v \times (w \times v) = (v^t v) w - (w^t v) v$$

5. Show that $O(a, v)$ (conjugation by (a, v) on the quaternions of the form $(0, w)$) is a rotation in the plane perpendicular to v by an angle that is determined by a and (v, v).

Solution: First of all, we note that conjugation preserves the norm of $(0, w)$. Hence $O(a, v)$ is orthogonal.

To simplify, we first calculate

$$(a^2 + v^t v) O(a, v) w = (a, v) \odot (0, w) \odot (a, -v)$$

We start with

$$(0, w) \odot (a, -v) = (v^t w, aw - w \times v)$$

Hence

$$(a,v)\odot(0,w)\odot(a,-v) = \\ \big(av^tw - v^t(aw - w\times v), a(aw - w\times v) + (v^tw)v + v\times(aw - w\times v)\big)$$

The right hand simplifies (using $v^t(w\times v) = 0$ (since the cross-product is perpendicular to each vector and $w\times v = -v\times w$) to

$$\big(0, a^2w + 2av\times w + (v^tw)v - v\times(w\times v)\big)$$

We can now use the previous exercise to further simplify this to get

$$O(a,v)w = \frac{a^2w + 2av\times w + 2(v^tw)v - (v^tv)w}{a^2 + v^tv}$$

Note that $O(a,v) = 1$ for $v = 0$. So we can assume that $v \neq 0$. Applying the above to v we get

$$O(a,v)v = \frac{a^2v + (v^tv)v}{a^2 + v^tv} = v$$

Hence, v is fixed by $O(a,v)$.

We have already noted that $O(a,v)$ is orthogonal. Hence, we only need to compute its action on vectors perpendicular to v.

Now suppose that w is a unit vector perpendicular to v. Then $v\times w$ is perpendicular to v and to w; further $(v\times w)^t(v\times w) = v^tv$. We thus see that

$$O(a,v)w = \frac{a^2w + 2av\times w - (v^tv)w}{a^2 + v^tv} = cw + d\frac{v\times w}{\|v\|}$$

where

$$c = \frac{a^2 - v^tv}{a^2 + v^tv}$$
$$d = \frac{2a\|v\|}{a^2 + v^tv}$$

We see that $c^2 + d^2 = 1$. In fact, if we put

$$c_1 = \frac{a}{\sqrt{a^2 + v^tv}}$$
$$d_1 = \frac{\|v\|}{\sqrt{a^2 + v^tv}}$$

Then $c_1^2 + d_1^2 = 1$ and, we have $c = c_1^2 - d_1^2$ and $d = 2c_1d_1$. This shows that if we take t so that $(\cos t, \sin t) = (c_1, d_1)$, then $O(a,v)$ rotates w by $2t$.

6. Show that $(a, v) \mapsto O(a, v)$ gives a group homomorphism from S^3 to the group $SO(3)$ of 3 dimensional rotations.

Solution: We have seen above that $O(a, v)$ is a rotation hence it is of determinant 1. We easily check that

$$(0, O(q_1 \odot q_2)w) = \\ (q_1 \odot q_2) \odot (0, w) \odot (q_1 \odot q_2)^{-1} = \\ q_1 \odot \left(q_2 \odot (0, w) \odot q_2^{-1}\right) \odot q_1^{-1} = \\ (0, O(q_1)O(q_2)w)$$

7. Show that this homomorphism is onto and has kernel $\{\pm 1\}$.

Solution: By the formula given in the previous exercise, we see that if $q = (\cos t, \sin tv)$ then $O(q)$ is rotation by $2t$ in the plane perpendicular to v. It follows that the map is onto as we vary v. Moreover, $O(q)$ is identity if and only if $2t = 2\pi$ or $2t = 0$. In the first case we have $q = (-1, 0)$ and in the second case we have $q = (1, 0)$.