Solutions to Assignment 11

1. What are some values of b and c that will make the following an orthogonal matrix?

$$G = \begin{pmatrix} 3/5 & b \\ c & 3/5 \end{pmatrix}$$

Solution: The condition that the columns form an orthonormal basis translates into the equations

$$(3/5)^2 + c^2 = 1 = b^2 + (3/5)^2 \text{ and } (3/5)b + c(3/5) = 0$$

This gives us $b^2 = c^2 = 16/25$ and $b = -c$. One solution is $b = 4/5$ and $c = -4/5$.

2. For the above matrix (after substituting b and c) calculate the matrix $H = (G-1)(G+1)^{-1}$. Check that it is skew-symmetric.

Solution: We calculate

$$G + 1 = \begin{pmatrix} 8/5 & 4/5 \\ -4/5 & 8/5 \end{pmatrix}$$

Hence, its inverse is

$$(G+1)^{-1} = \frac{5}{16}\begin{pmatrix} 8/5 & -4/5 \\ 4/5 & 8/5 \end{pmatrix}$$

So we get

$$H = (G-1)(G+1)^{-1} = \frac{5}{16}\begin{pmatrix} -2/5 & 4/5 \\ -4/5 & -2/5 \end{pmatrix}\begin{pmatrix} 8/5 & -4/5 \\ 4/5 & 8/5 \end{pmatrix}$$

We calculate this to be

$$H = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$$

Which is skew symmetric.

3. Let $v = e_1 + e_2 + e_3$ be the column 3-vector all of whose entries are 1. Write down the matrix of the associated reflection R_v defined as the linear transformation

$$R_v(w) = w - 2\frac{w^t v}{v^t v} v$$

Solution: We calculate $v^t v = 3$ and $e_i^t v = 1$ for all i. Hence,

$$R_v(e_i) = e_i - (2/3)v$$

Hence, the matrix of the linear transformation is

$$\begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}$$

4. More generally, if v is *any* column vector, then show that the matrix for R_v is given by

$$R_v = 1 - \frac{2}{v \cdot v} v v^t$$

(Note that vv^t is a 3×3 matrix!)

Solution: We note that $vv^t w = (v^t w)v$ since $v^t w$ is just a scalar. Moreover, $v^t w = w^t v$. It follows that

$$\left(1 - \frac{2}{v^t v} v v^t\right) w = w - 2\frac{w^t v}{v^t v} v$$

as required.

5. Using the above formula, write $A = R_v R_u$ in a form where it is "obvious" that A is identity on the vectors perpendicular to both u and v. Calculate the 2×2 matrix for A on the plane spanned by u and v (assume that u and v are linearly independent). Is it an orthogonal matrix? If not, why not?

Solution: Using the above formula we have

$$A = 1 - \frac{2}{v^t v} v v^t - \frac{2}{u^t u} u u^t + 4\frac{v^t u}{(v^t v)(u^t u)} v u^t$$

If w is perpendicular to both v and u then $v^t w = u^t w = 0$ and so it is obvious that $Aw = w$. We can thus restrict our attention to the space spanned by v and u. We have

$$Av = v - 2v - 2\frac{u^t v}{u^t u} u + 4\frac{(v^t u)^2}{(v^t v)(u^t u)} v = \left(-1 + 4\frac{(v^t u)^2}{(v^t v)(u^t u)}\right) v - 2\frac{v^t u}{u^t u} u$$

similarly,

$$Au = u - 2\frac{v^t u}{v^t v} v - 2u + 4\frac{v^t u}{(v^t v)} v = 2\frac{v^t u}{v^t v} v - u$$

This gives us the matrix of A restricted to the space spanned by u and v in terms of the basis u and v as

$$\begin{pmatrix} \left(-1 + 4\frac{(v^t u)^2}{(v^t v)(u^t u)}\right) & 2\frac{v^t u}{u^t u} \\ 2\frac{v^t u}{v^t v} & -1 \end{pmatrix}$$

We see that this has determinant 1. However, it is not orthogonal unless $v^t u = 0$, in which case it is the matrix of multiplication by -1. (In other words, if v and u are perpendicular to each other, then it is a rotation by π radians.) In the other cases this *does* represent a rotation but in a non-orthogonal basis u and v; this is why it is not an orthogonal matrix.

6. For the quaternion $q = (1, (1, 1, 1))/2$ carry out the following:

   (a) Check that the norm of q is 1.

   Solution: We see that $\mathrm{Nm}(1, (1, 1, 1)) = 1 + 3 = 4 = 2^2$. Hence $\mathrm{Nm}(q) = 1$.

   (b) Write down the 4×4 matrix A of the linear transformation $(b, w) \mapsto q \odot (b, w)$.

   Solution: We have $(1, 0) \mapsto q \odot (1, 0) = q$ and

   $$(0, e_i) \mapsto q \odot (0, e_i) = (1/2)(-1, e_i + (1, 1, 1) \times e_i)$$

   Now we have

   $$\begin{aligned} (e_1 + e_2 + e_3) \times e_1 &= e_2 - e_3 \\ (e_1 + e_2 + e_3) \times e_2 &= e_3 - e_1 \\ (e_1 + e_2 + e_3) \times e_3 &= e_1 - e_2 \end{aligned}$$

   Hence, the matrix is

   $$A = \begin{pmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \end{pmatrix}$$

   (c) Write down the 3×3 matrix B of the linear transformation $(0, w) \mapsto q \odot (0, w) \odot \overline{q}$.

Solution: We have already calculated that

$$
\begin{aligned}
q \odot (0, e_1) &= (-1, e_1 + e_2 - e_3)/2 \\
q \odot (0, e_2) &= (-1, e_2 + e_3 - e_1)/2 \\
q \odot (0, e_3) &= (-1, e_3 + e_1 - e_2)/2
\end{aligned}
$$

Now $\overline{q \odot v} = \overline{v} \odot \overline{q}$ and $\overline{e_i} = -e_i$. So we get

$$
\begin{aligned}
q \odot (0, e_1) \odot \overline{q} &= (1/2)\left(-\overline{q} - \overline{q \odot e_1} - \overline{q \odot e_2} + \overline{q \odot e_3}\right) \\
q \odot (0, e_2) \odot \overline{q} &= (1/2)\left(-\overline{q} + \overline{q \odot e_1} - \overline{q \odot e_2} - \overline{q \odot e_3}\right) \\
q \odot (0, e_3) \odot \overline{q} &= (1/2)\left(-\overline{q} - \overline{q \odot e_1} + \overline{q \odot e_2} - \overline{q \odot e_3}\right)
\end{aligned}
$$

Substituting once more and taking conjugates, we get

$$
\begin{aligned}
q \odot (0, e_1) \odot \overline{q} &= (0, e_2) \\
q \odot (0, e_2) \odot \overline{q} &= (0, e_3) \\
q \odot (0, e_3) \odot \overline{q} &= (0, e_1)
\end{aligned}
$$

This gives the matrix

$$
B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
$$

(d) Are A and B orthogonal matrices?

Solution: Both A and B are orthogonal matrices. This is because $\mathrm{Nm}(a, v)$ is the natural inner-product on 4 dimensional space and the above multiplications preserve the norm (since $\mathrm{Nm}(q) = 1$).

(e) Describe the matrix B as a rotation. What is the angle in terms of sine and cosine?

Solution: This is a rotation by $2\pi/3$ in the plane perpendicular to the vector $(1, 1, 1)$.

(f) Can you describe the matrix A as a rotation (is there any fixed "axis")?

Solution: If $q \odot (b, w) = (b, w)$ for some non-zero (b, w), then we can multiply on the right by the *inverse* of (b, w) to get $q = 1$. Thus, no vector can be fixed by mutliplication by a quaternion q, unless the quaternion is 1.

(g) What is the canonical form of B over complex numbers?

Solution: Since B is a rotation by $2\pi/3$ in a suitable plane, the canonical form is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi\sqrt{-1}/3} & 0 \\ 0 & 0 & e^{-2\pi\sqrt{-1}/3} \end{pmatrix}$$

7. Fix a 3-vector v. Write the matrix for the linear transformation $w \mapsto v \times w$.

Solution: We let the coordinates of v to be x, y and z. Then we get

$$\begin{aligned} e_1 \mapsto & -ye_3 + ze_2 \\ e_2 \mapsto & xe_3 - ze_1 \\ e_3 \mapsto & -xe_2 + ye_1 \end{aligned}$$

Hence the matrix is

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

In particular, note that it is a skew-symmetric matrix and that every skew-symmetric matrix is also obtained this way!

8. Repeat the above calculations for a *general* unit quaternion $q = (a, v)$ with $a^2 + v \cdot v = 1$.

Solution: By calculations similar to the one above, we see that the matrix associated with $(b, w) \mapsto q \odot (b, w)$, where $q = (a, v)$ and $v = (x, y, z)$ (assuming $a^2+x^2+y^2+z^2 = 1$) is given by

$$A(q) = \begin{pmatrix} a & -x & -y & -z \\ x & a & -z & y \\ y & z & a & -x \\ z & -y & x & a \end{pmatrix}$$

As above, to calculate the matrix of

$$(0, w) \mapsto q \odot (0, w) \odot \overline{q}$$

we observe that $(c,u)\overline{q} = \overline{q \odot (c,-u)}$. Now, the conjugate of a quaternion, written as a column vector is given by multiplication by the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

So we need to calculate $C \cdot A(q) \cdot C \cdot A(q)$ in other words

$$C \cdot \begin{pmatrix} a & -x & -y & -z \\ x & a & -z & y \\ y & z & a & -x \\ z & -y & x & a \end{pmatrix} \begin{pmatrix} a & -x & -y & -z \\ -x & -a & z & -y \\ -y & -z & -a & x \\ -z & y & -x & -a \end{pmatrix}$$

We get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2+x^2-z^2-y^2 & 2(xy-az) & 2(xz+ay) \\ 0 & 2(xy+az) & a^2+y^2-z^2-x^2 & 2(yz-ax) \\ 0 & 2(xz-ay) & 2(yz+ax) & a^2+z^2-y^2-x^2 \end{pmatrix}$$

So the 3×3 matrix is

$$\begin{pmatrix} a^2+x^2-z^2-y^2 & 2(xy-az) & 2(xz+ay) \\ 2(xy+az) & a^2+y^2-z^2-x^2 & 2(yz-ax) \\ 2(xz-ay) & 2(yz+ax) & a^2+z^2-y^2-x^2 \end{pmatrix}$$

We check that

$$\begin{pmatrix} a^2+x^2-z^2-y^2 & 2(xy-az) & 2(xz+ay) \\ 2(xy+az) & a^2+y^2-z^2-x^2 & 2(yz-ax) \\ 2(xz-ay) & 2(yz+ax) & a^2+z^2-y^2-x^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where we used the identity $a^2+x^2+y^2+z^2 = 1$. It follows that this is a rotation in the plane perpendicular to the vector (x,y,z).