Solutions to Assignment 9

1. Write down the quadratic form associated with the symmetric bilinear form $A(v, w) = v^t A w$, where A is the 3×3 matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Solution: We calculate

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + 2xy + y^2 + 2yz$$

2. Write down the symmetric 3×3 matrix A associated with the quadratic form $Q(x, y, z)$ given by

$$Q(x, y, z) = xy + yz + z^2$$

Solution: We calculate

$$Q(x+x', y+y', z+z') = (xy+x'y'+xy'+x'y)+(yz+y'z'+yz'+y'z)+(z^2+z'^2+2zz')$$

Thus $Q(x + x', y + y', z + z') - Q(x, y, z) - Q(x', y', z')$ is

$$B((x, y, z), (x', y', z')) = xy' + x'y + yz' + y'z + 2zz' = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

3. Use completion of the square to write each quadratic form below in diagonal form.

   (a) $x^2 + 2xy + 2xz + 2y^2 + 3yz + 3z^2$

   Solution: We calculate

$$x^2 + 2xy + 2xz = (x + y + z)^2 - y^- 2yz - z^2$$

Hence,

$$x^2 + 2xy + 2xz + 2y^2 + 3yz + 3z^2 = (x + y + z)^2 + y^2 + yz + 2z^2$$

Now,

$$y^2 + yz = (y + z/2)^2 - z^2/4$$

So,

$$x^2 + 2xy + 2xz + 2y^2 + 3yz + 3z^2 = (x + y + z)^2 + (y + z/2)^2 + (7/4)z^2$$

(b) $x^2 + 2xy + 2xz + y^2 + yz + 2z^2$

Solution: As above we have

$$x^2 + 2xy + 2xz + y^2 + yz + 2z^2 = (x + y + z)^2 - yz + z^2$$

Now,

$$z^2 - yz = (z - y/2)^2 - y^2/4$$

So,

$$x^2 + 2xy + 2xz + y^2 + yz + 2z^2 = (x + y + z)^2 + (z - y/2)^2 - y^2/4$$

4. In each of the examples in the question above write the associated symmetric matrix A that gives the bilinear form. Moreover, write the change of co-ordinates matrix S so that S^tAS is in diagonal form.

Solution: As before, we easily calculate the symmetric matrix associated with $x^2 + 2xy + 2xz + 2y^2 + 3yz + 3z^2$ to be

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3/2 \\ 1 & 3/2 & 3 \end{pmatrix}$$

and that associated with $x^2 + 2xy + 2xz + y^2 + yz + 2z^2$ to be

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1/2 \\ 1 & 1/2 & 2 \end{pmatrix}$$

We saw above that the change of coordinates to get a diagonal form in the first case was given by $u = x+y+z$, $v = y+z/2$ and $w = z$. The inverse change of co-ordinates is given by $z = w$, $y = v - w/2$ and $x = u - v - w/2$. Thus, we see that

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1/2 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3/2 \\ 1 & 3/2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7/4 \end{pmatrix}$$

We saw above that the change of coordinates to get a diagonal form in the second case was given by $u = x + y + z$, $v = y$, $w = z - y/2$. The inverse change of co-ordinates is given by $y = v$, $z = w + v/2$ and $x = u - (3/2)v - w$. Thus we see that

$$\begin{pmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 1/2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1/2 \\ 1 & 1/2 & 2 \end{pmatrix} \begin{pmatrix} 1 & -3/2 & -1 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5. Find the rank and signature of the following symmetric matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Solution: The associated quadratic form is

$$w^2 + 2wx + 2wz + 2xy + 2xz + z^2$$

We simplify

$$\begin{aligned} w^2 + 2wx + 2wz + 2xy + 2xz + z^2 &= \\ (w + z)^2 + 2x(w + z) + 2xy &= \\ (w + z + x)^2 - x^2 + 2xy &= \\ &(w + x + z)^2 - (x - y)^2 + y^2 \end{aligned}$$

Since the variables $w + x + z$, $x - y$ amd y are independent, we see that the rank is 3 and the signature is $(2, 1, 1)$ (i.e. 2 positive, 1 null and 1 negative).

6. Let v_1, v_2, v_3 be the column vectors of the matrix below. Apply Gram-Schmidt orthogonalisation to find a basis of orthogonal vectors of the form v_1, $v_2 + av_1$, $v_3 + bv_1 + cv_2$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution: We have $w_2 = v_2 + av_1$ so that w_2 is orthogonal to $w_1 = v_1$ so $a = -(w_1^t v_2)/(w_1^t w_1) = -1/2$. Hence, we get

$$w_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

Similarly, $w_3 = v_3 + bv_1 + cv_2$ so that w_3 is orthogonal to w_1 and w_2. We write $w_3 = v_3 + b'w_1 + c'w_2$ where

$$b' = -(w_1^t v_3)/(w_1^t w_1) = -1/2 \text{ and } c' = -(w_2^t v_3)/(w_2^t w_2) = -(1/2)/(3/2) = -1/3$$

Hence, we get

$$w_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

The final orthogonal matrix is

$$\begin{pmatrix} 1 & -1/2 & 2/3 \\ 1 & 1/2 & -2/3 \\ 0 & 1 & 2/3 \end{pmatrix}$$

7. Given a symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Write down the conditions on the variables a, b and c that will make this a positive definite matrix.

Solution: We can write the associated quadratic form $ax^2 + 2bxy + cy^2$. Since it has to be positive definite, we see that $a > 0$ since the form has to be positive on $(1, 0)$. By completing the square, we get $a(x + by/a)^2 + (c - b^2/a)y^2$. This shows that $c - b^2/a > 0$ or equivalently $ac > b^2$. Since the above form is diagonal, we see that these two conditions $a > 0$ and $ac > b^2$ are sufficient as well.

8. (Starred) Find a similar condition for the 6 entries of a symmetric 3×3 matrix.

Solution: Let us write the quadratic form for a symmetric 3×3 matrix

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

with variable coefficients

$$ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$$

We complete the first square to get

$$a(x + by/a + cz/a)^2 + (d - b^2/a)y^2 + 2(e - bc/a)yz + (f - c^2/a)z^2$$

As above, this means that $d - b^2/a > 0$, or equivalently $ad > b^2$ for positivity. Further completing the square for the last three terms

$$(d - b^2/a)\left(y + \frac{ae - bc}{ad - b^2}z\right)^2 + \left(\frac{af - c^2}{a} - \frac{(ae - bc)^2}{a(ad - b^2)}\right)z^2$$

This gives the inequality

$$(af - c^2)(ad - b^2) > (ae - bc)^2$$

Simplifying this and using $a > 0$, we get

$$afd - fb^2 - dc^2 - ae^2 + 2ebc > 0$$

Grouping terms together, this becomes

$$a(df - e^2) - b(bf - ce) + c(be - dc) > 0$$

We note that the left-hand side is the determinant of the 3×3 matrix above. So we see that positivity is equivalent to the statement that the 1×1, 2×2 and 3×3 matrices along the diagonal are positive! This is not an accident, it can be shown, with a little work that this is the case for $n \times n$ matrices.

9. Given a symmetric matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

   (a) Find a vector v (with real entries) with $v \cdot v = 1$ for which the quadratic form takes a maximum value.

   Solution: Putting $v = (x, y)$ we see that the quadratic form is $q(x, y) = x^2 + 2xy$. Since the maximum is taken at an eigenvector, we calculate $\det(T - A) = T^2 - T - 1$ with solution $T = \phi = (1 + \sqrt{5})/2$ as the largest. The corresponding eigenvector is $(\phi, 1)$ and so the unit eigenvector is $(\phi, 1)/\sqrt{2 + \phi}$.

   (b) Using the above vector write an orthonormal matrix S so that $S^{-1}AS$ is diagonal.

Solution: The second eigenvector is perpendicular to the above one. Hence, it is $(-1, \phi)/\sqrt{2+\phi}$ with eigen value $1-\phi$. It follows that we put

$$S = \frac{1}{\sqrt{2+\phi}} \begin{pmatrix} \phi & -1 \\ 1 & \phi \end{pmatrix}$$

to get $S^t AS$ as a digonal matrix with entries ϕ and $1-\phi$.

10. (Starred) Given a symmetric matrix A over $\mathbb{Q}$, show that its minimal polynomial has distinct roots.

Solution: A symmstric matrix A over $\mathbb{Q}$ is also a symmetric matrix over $\mathbb{R}$. Hence, it is diagonalisable over $\mathbb{R}$. It follows that its minimal polynomial over $\mathbb{Q}$ has distinct roots.

11. (Starred) If a symmetric matrix A is also nilpotent, then show that A is the 0 matrix.

Solution: Since the matrix A is diagonalisable and nilpotent it is 0.