Solutions to Quiz 5

1. Given the symmetric matrix below:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(a) Write down the quadratic form associated with the matrix.

Solution: We calculate

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2(xy + yz + zx)$$

(b) Convert the quadratic form to diagonal form.

Solution: We calculate

$$2xy = \frac{1}{2}\left((x+y)^2 - (x-y)^2\right)$$

Using this we get

$$2xy + 2yz + 2zx = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2 + 2z(x+y)$$

Next we see that

$$\frac{1}{2}(x+y)^2 + 2z(x+y) = \frac{1}{2}\left((x+y)^2 + 4z(x+y)\right) = \frac{1}{2}\left((x+y+2z)^2 - 4z^2\right)$$

Substituting this we get

$$2xy + 2yz + 2zx = \frac{1}{2}(x+y+2z)^2 - 2z^2 - \frac{1}{2}(x-y)^2$$

Thus with the new variables $u = x + y + 2z$, $v = z$ and $w = x - y$ we have the diagonal form

$$\frac{1}{2}u^2 - 2v^2 - \frac{1}{2}w^2$$

(c) Write down the *minimal* polynomial of the matrix.

Solution: We calculate the characteristic polynomial

$$\det\begin{pmatrix} T & -1 & -1 \\ -1 & T & -1 \\ -1 & -1 & T \end{pmatrix} = \\ T(T^2-1)+(-T-1)-(1+T) = (T+1)(T^2-T-2) = \\ (T+1)^2(T-2)$$

Since the matrix is diagonalisable, its minimal polynomial has distinct roots *and* all the roots of the characteristic polynomial are roots of it. Hence, the minimal polynomial is $(T+1)(T-2)$.