Complex Matrices

1. Given a complex inner product space V and a linear transformation $A : V \to V$. Suppose B is a *set-theoretic* map such that $\langle Av, w\rangle = \langle v, Bw\rangle$ for all v, w in V. Is it true that B is a linear map?

2. Consider the linear transformation $A : \mathbb{C}^2 \to \mathbb{C}^2$ given by interchanging the two coordinates. Is it unitary? Is it self-adjoint? What is the matrix M of this linear transformation in the bases $e_1 = (1, 0)$, $e_2 = (1, 1)$. Is M a unitary or Hermitian matrix? Why not?

3. Write down the linear transformation that is an orthogonal projection to the subspace generated by $(1, -1, 0)$ and $(1, 0, -1)$.

4. Classify the following matrices as unitary, Hermitian, idempotent, orthogonal idempotent, normal or none of these.

   (a)
   $$\begin{pmatrix} 0 & \iota \\ -\iota & 0 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

   (c)
   $$\begin{pmatrix} 1 & \iota \\ -\iota & 1 \end{pmatrix}$$

   (d)
   $$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

   (e)
   $$\begin{pmatrix} \iota\cos t & \iota\sin t \\ -\iota\sin t & \iota\cos t \end{pmatrix}$$

   (f)
   $$\begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix}$$

5. Consider the 2×2 invertible matrix
   $$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

   Write the KAN, KP and KAK decompositions for this matrix. Are the resulting matrices real or complex?

6. Consider the 2×2 invertible matrix

$$\begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

Write the KAN, KP and KAK decompositions for this matrix. Are the resulting matrices real or complex?

7. (Starred) Will it always be true that for real matrices the KAN, KP and KAK decompositions will give real matrices?

8. Check that the following matrix is normal.

$$\begin{pmatrix} 1 & -\iota \\ -\iota & 1 \end{pmatrix}$$

Find the spectral decomposition of this matrix. (Equivalently, find an orthonormal basis of eigenvectors.)