Spectral Theorem for Normal Operators

Recall that a normal operator S is one that commutes with its own adjoint S^*.

We will work over finite dimensional vector spaces.

We have seen earlier that an invertible normal operator G can be written as $G = KP = PK$ where K is unitary and P is positive self-adjoint.

We have also seen that a unitary operators (respectively positive self-adjoint operator) has an orthonormal basis of eigenvectors.

Simultaneous Diagonalisation

Suppose A and B are commuting matrices. Then, for an eigenvector v of A we have

$$ABv = BAv = B\lambda v = \lambda Bv$$

where λ is the eigenvalue of A associated with the eigenvector v. Thus, if we denote by V_λ the space of all vectors v (including 0!) for which $Av = \lambda v$, then B takes this subspace to itself.

If A is diagonalisable, then we can write a basis of eigenvectors, hence we can express every vector v as a (unique) linear combination of elements $v_\lambda \in V_\lambda$ as we vary λ over eigenvalues of A. Further B takes each V_λ to itself.

If, in addition, B is diagonalisable, then we can further write V_λ in terms of a basis of eigenvectors for B. (Note that the minimal polynomial of $B_{|V_\lambda}$ divides the minimal polynomial of B and thus has distinct roots!). This gives a basis of our original vector space which consists of *simultaneous* eigenvectors for both A and B (since each non-zero element of V_λ is an eigenvector of A).

In summary, if A and B commute and are diagonalisable, then we can find a basis consisting of simultanous eigenvectors for A and B. This argument can easily be extended to any set of commuting diagonalisable matrices.

The invertible case

We now apply this to the decomposition $G = KP = PK$ for a normal invertible matrix G which we saw above. Recall that K is unitary and P is positive definite. By the above argument, we can find an *unitary* basis consisting of simultaneous eigenvectors for K and P. It is clear that these are eigenvectors for G as well.

Let $V_{k,p}$ denote the subspace of vectors v so that $Kv = kv$ and $Pv = pv$; in other words, $V_{k,p}$ is the subspace spanned by simultanous eigenvectors for K and P with eigenvalues k and p respectively. Let v be in $V_{k,p}$ and w be in $V_{k',p'}$ with $(k,p) \neq (k',p')$. We then calculate

$$p\langle v, w\rangle = \rangle Pv, w\rangle = \langle v, Pw\rangle = p'\langle v, w\rangle$$

where we have used the fact that p' is real. If $p \neq p'$, then this says $\langle v, w\rangle = 0$. Similarly,

$$\langle v, w\rangle = \rangle Uv, Uw\rangle = k\overline{k'}\langle v, w\rangle$$

Now, k' is a complex number of norm 1 so $\overline{k'}$ its multiplicative inverse. Thus, if $k \neq k'$, then $\langle v, w\rangle = 0$. It follows that $V_{k',p'}$ is contained in $V_{k,p}^{\perp}$ if $(k,p) \neq (k', p')$.

Let $E_{k,p}$ be the orthogonal projection onto $V_{k,p}$ that was constructed earlier. We see that $E_{k,p}E_{k',p'} = 0$ if $(k,p) \neq (k',p')$. Moreover,

$$K = \sum_k k \sum_p E_{k,p} \text{and} P = \sum_p p \sum_k E_{k,p}$$

since both K and P are diagonalisable. It follows that

$$G = KP = \sum_{k,p} kpE_{k,p}$$

Now note that $\lambda = kp$ is an eigenvalue of G. Moreover, since p is positive, we have $|\lambda| = p$. So, $\lambda = \lambda'$, if and only if $(k,p) = (k'p')$.

In other words, G is a sum of matrices of the form λE_λ where λ is an eigenvalue of G and E_λ is an orthogonal projection onto the subspace where G acts as multiplication by λ (this is called the λ-eigenspace of G). Moreover, $E_\lambda E_\mu = 0$ if $\lambda \neq \mu$.

Spectral Theorem for Normal Operators

Now suppose that M is a normal operator, i.e. M commutes with M^*, on a finite dimensional complex inner-product space V. As seen earlier, V has an orthonormal basis. Let M also denote the matrix of of M with respect to this bases. As seen earlier the matrix of the adjoint operator is also M^* with respect to this orthonormal basis.

Consider the characteristic polynomial $P(T) = \det(M - T \cdot 1)$. This is a non-zero polynomial since its leading coefficient T^n where n is the dimension of V. Hence there is an integer k so that $P(k) \neq 0$. It follows that $G = M - k \cdot 1$ is an invertible matrix. Clearly $G^* = M^* - k \cdot 1$ commutes with M and thus also with G. In other words G is a normal invertible matrix.

By what has been proved above we have an expression $G = \sum_\lambda \lambda E_\lambda$ where E_λ are orthogonal projectors such that $E_\lambda E_\mu = 0$ if $\lambda \neq \mu$. As proved above G has a basis consisting of eigenvectors each such vector is in the image of E_λ for some λ (note that G has only non-zero eigenvalues!). It follows that $E = \sum_\lambda E_\lambda$ is identity on this basis of eigenvectors; hence, $E = 1$. We thus obtain the identity $M = G + k \cdot 1 = \sum_\lambda (\lambda + k)E_\lambda$. We have thus obtained “spectral decomposition” of the normal operator M.

Conversely, suppose $M = \sum_\lambda \lambda E_\lambda$ where E_λ are orthogonal idempotents with $E_\lambda E_\mu = 0$ if $\lambda \neq \mu$. Then $M^* = \sum_\lambda E_\lambda$, since $E_\lambda^* = E_\lambda$. Moreover, M commutes with all the E_λ, and thus commutes with M^* as well.

In summary, a matrix is normal *if* and *only if* it can be written as a linear combination of orthogonal idempotents E_λ such that $E_\lambda E_\mu = 0$ when $\lambda \neq \mu$ and $\sum_\lambda E_\lambda = 1$. This is called the *spectral* theorem for normal operators and the above expression for the operator is called the spectral decomposition of the operator.