Matrix Decompositions

The "pivot" method for doing row reduction of a matrix M can be seen as writing $M = \pi LU$ where π is a permutation matrix, L is lower triangular (with no restriction on the diagonal entries) and U is upper triangular matrix with 1's on the diagonal.

There are other ways in we can decompose matrices. In what follows, we will assume that G is an invertible matrix with complex entries.

KAN

The columns of G can be seen as a basis $v_1, \ldots, v_n$ of the vector space $\mathbb{C}^n$ which is an inner product space with the usual inner product $\langle v, w\rangle = \overline{w}^t v$ (using the column vector notation for elements of $\mathbb{C}^n$). As seen above, we can apply Gram-Schmidt orthogonalisation to obtain a new basis $f_1, \ldots, f_n$ so that

$$[f_1, \ldots, f_n] = [v_1, \ldots, v_n]N_1$$

where N_1 is an upper triangular matrix with 1's on the diagonals. Further, we can write $e_i = f_i/|f_i|$ where $|f_i|$ is the positive square root of (the positive real number) $\langle f_i, f_i\rangle$.

$$[e_1, \ldots, e_n] = [f_1, \ldots, f_n]A_1$$

where A_1 is a diagonal matrix with diagonal entries $1/|f_i|$ which are positive real numbers. Now $e_1, \ldots, e_n$ is an orthonormal basis so that $K = [e_1, \ldots, e_n]$ is a unitary matrix. We put $N = N_1^{-1}$ and $A = A_1^{-1}$. (Note that N is also upper triangular with with 1's on the diagonal). It follows that we have

$$G = [v_1, \ldots, v_n] = K \cdot A \cdot N$$

where K is a unitary matrix, A is a diagonal matrix with positive real numbers on the diagonal and N is a upper triangular matrix with 1's on the diagonal.

KP

Consider the matrix $Q = G^*G$. We easily check that this is Hermitian. In fact it also satisfies $\overline{v}^t Qv \geq 0$ with equality if and only if $v = 0$; in other words, it is positive definite.

In general, if $A : V \to V$ is a self-adjoint operator on a complex inner-product space, we have seen that $\langle Av, v\rangle$ is a real number. If *in addition* it is a non-negative real number which is 0 if and only if $v = 0$, we say that A is a positive (definite) self-adjoint operator.

We have seen that there is an orthonormal basis $u_1, \ldots, u_n$ which consists of eigenvectors of Q. We put $U = [u_1, \ldots, u_n]$ so that we have $QU = UB$ where

B is a diagonal matrix with diagonal entries as the eigenvalues of Q. We easily show that the eigenvalues of Q are *positive* real numbers since Q is positive self-adjoint. It follows that $B = A^2$ where A is the diagonal matrix with diagonal entries as the positive square roots of the corresponding diagonal entries of B. If we put $P = UAU^{-1}$, then we easily check that P is Hermitian and self-adjoint and that $P^2 = Q$. Finally, we put $K = GP^{-1}$ so that $G = KP$. We note that

$$K^*K = (GP^{-1})^*GP^{-1} = P^{-1}G * GP^{-1} = P^{-1}QP^{-1} = 1$$

where we have used $P^* = P$ and $P^2 = Q$. In other words, K is unitary.

It follows that any invertible matrix can be written as the product of a unitary matrix and a positive definite matrix.

For future reference, let us note that if G is a *normal* matrix (i.e. G^* commutes with G), then G and Q commute. It follows easily that G and P also commute and hence K and P also commute. In other words, a normal matrix can be written as a product of a *commuting* pair of a unitary matrix and a positive definite matrix.

KAK

As above, we can write $Q = G^*G$, which is a positive definite Hermitian matrix. Repeating the above, we can write $QU = UB$, where U is a unitary matrix and B a diagonal matrix with positive entries on the diagonal. As before, we put A as the square-root of B obtained as a diagonal matrix with diagonal entries as the square-root of the corresponding diagonal entries of B. We then put $V = GUA^{-1}$ and calculate

$$V^*V = A^{-1}U^*G^*GUA^{-1} = A^{-1}U^*QUA^{-1} = A^-1BA^{-1} = 1$$

where we have used $A^* = A$ and $U^* = U^{-1}$. Thus, we have written $G = VAU^*$. In other words, every invertible matrix is a product of a unitary matrix, a diagonal matrix with positive real entries on the diagonal, followed by another unitary matrix.

All of the above decompositions are useful when we want to solve various computational problems connected with matrices.