Complex Inner Product Spaces

A vector space V over the field $\mathbb{C}$ of complex numbers is called a (complex) *inner product space* if it is equipped with a pairing

$$\langle , \rangle : V \times V \to \mathbb{C}$$

which has the following properties:

1. (Sesqui-symmetry) For every v, w in V we have $\langle w, v\rangle = \overline{\langle v, w\rangle}$.
2. (Linearity) For every u, v, w in V and a in $\mathbb{C}$ we have $langleau + v, w\rangle = a\langle u, w\rangle + \langle v, w\rangle$.
3. (Positive definite-ness) For every v in V, we have $\langle v, v\rangle \geq 0$. Moreover, if $\langle v, v\rangle = 0$, the v must be 0.

Sometimes the third condition is replaced by the following weaker condition:

3'. (Non-degeneracy) For every v in V, there is a w in V so that $\langle v, w\rangle \neq 0$.

We will generally work with the stronger condition of positive definite-ness as is conventional.

A basis $f_1, f_2, \ldots$ of V is called orthogonal if $\langle f_i, f_j\rangle = 0$ if $i < j$. An orthogonal basis of V is called an *orthonormal* (or *unitary*) basis if, in addition $\langle f_i, f_i\rangle = 1$.

Given a linear transformation $A : V \to V$ and another linear transformation $B : V \to V$, we say that B is the adjoint of A, if for all v, w in V, we have $\langle Av, w\rangle = \langle v, Bw\rangle$. By sesqui-symmetry we see that A is then the adjoint of B as well. Moreover, by positive definite-ness, we see that if A has an adjoint B, then this adjoint is *unique*. Note that, when V is infinite dimensional, we may not be able to find an adjoint for A! In case it does exist it is denoted as A^*.

A linear transformation $A : V \to V$ is called a *self-adjoint operator* on V (with respect to $\langle , \rangle$) if we have, for all v and w in V,

$$\langle Av, w\rangle = \langle v, Aw\rangle$$

Comparing with the definition above, this is equivalent to the assertion that A is its own adjoint.

An *invertible* linear transformation $U : V \to V$ is called a *unitary operator* on V (with respect to $\langle , \rangle$) if we have, for all v and w in V,

$$\langle Uv, Uw\rangle = \langle v, w\rangle$$

Since U has an inverse U^{-1}, we can also write this as

$$\langle Uv, w\rangle = \langle v, U^{-1}w\rangle$$

Comparison with the above identity shows that $U^* = U^{-1}$.

Given a linear transformation $T : V \to V$ which has an adjoint T^*. We say that T is *normal* if T^* commutes with T. Note that an operator commutes with itself and with its inverse (if the inverse exists). This shows that self-adjoint operators and unitary operators are normal.

Gram-Schmidt Orthogonalisation

Given a complex inner product space and a basis $v_1, v_2, \ldots v_n$, we would like to produce and orthogonal/orthonormal basis. The procedure is essentially identical to the earlier one for the real case.

We define $f_1 = v_1$ and put

$$w_k = v_k - \frac{\langle v_1, v_k \rangle}{\langle v_1, v_1 \rangle} v_1 \text{ for } k \geq 2$$

This makes $\langle f_1, w_k \rangle = 0$ for all $k \geq 2$. We can then repeat the process with $w_2, \ldots, w_n$. This results in a basis (**Exercise**: Why is it a basis?) $f_1, f_2, \ldots, f_n$, so that $\langle f_i, f_j \rangle = 0$ for $i < j$. In other words, this is an orthogonal basis.

Defining $e_i = f_i / \langle f_i, f_i \rangle^{1/2}$, we see that this is an orthonormal or unitary basis.

In particular, this shows that any finite dimensional complex inner product space has a orthonormal (unitary) basis.

Suppose $f : V \to \mathbb{C}$ is a linear map and let $f(e_i) = a_i$. We then define the vector $w_f = \sum_{i=1}^{n} \overline{a_i} e_i$; note that this sum *does not* make sense if V is infinite dimensional!

Now note that $\langle e_i, w_f \rangle = f(e_i)$. This can be used to show that $\langle v, w_f \rangle = f(v)$ for *all* vectors v in V since both sides are linear maps on V and they are equal on the basis vectors. Moreover, if $\langle v, w \rangle = \langle v, w' \rangle$ for all v, then it is easy to show that $w = w'$. Thus, we have shown that any linear map $f : V \to \mathbb{C}$ on a finite dimensional complex inner product space is of the form $v \mapsto \langle v, w \rangle$ for a suitably chosen vector w in V.

Given an orthonormal basis $e_1, \ldots, e_n$ of a complex inner product space, let us try to understand the inner product in terms of co-ordinates. Given vectors $v = \sum_{i=1}^{n} v_i e_i$ and $w = \sum_{i=1}^{n} w_j e_j$, we see that

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i \overline{w_i} = \begin{pmatrix} \overline{w_1} & \ldots & \overline{w_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Note that this expression will *not* be correct if we use a basis which is not orthonormal. Now if U is a unitary transformation, and we define $u_i = U e_i$, the $u_1, \ldots, u_n$ is another orthonormal basis. Writing $u_i = \sum_{j=1}^{n} u_{j,i} e_j$ as a column vector with entries $u_{i,j}$, and using the calculation of the inner-product above, we see that

$$\begin{pmatrix} \overline{u_{1,1}} & \dots & \overline{u_{n,1}} \\ \vdots & \ddots & \vdots \\ \overline{u_{n,1}} & \dots & \overline{u_{n,n}} \end{pmatrix} \cdot \begin{pmatrix} u_{1,1} & \dots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \dots & u_{n,n} \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$

In other words, the matrix M associated with a unitary transformation in an orthonormal basis satisfies $\overline{M}^t M = 1$, i.e. it is a unitary matrix. This is why an orthonormal basis is sometimes also called a unitary basis.

Similarly, let us examine the matrix associated with a self-adjoint transformation A. Writing $Ae_i = \sum_{i=1}^n a_{i,j} e_j$, we have (since $e_1, \dots, e_n$ is orthonormal)

$$a_{i,j} = \langle Ae_i, e_j \rangle = \langle e_i, Ae_j \rangle = \overline{\langle Ae_j, e_i \rangle} = \overline{a_{j,i}}$$

Thus the matrix associated with A is Hermitian, i.e. $A = \overline{A}^t$.

Now suppose that $A : V \to V$ is a linear transformation and for a vector w in V, let us define the linear functional $f = f_{A,w} : v \mapsto \langle Av, w \rangle$. When V is a finite dimensional complex inner product space, we have seen that there is a vector w_f so that $f(v) = \langle v, w_f \rangle$. Thus, we get a *set-theoretic* map $B : V \to V$ defined by $w \mapsto w_f$. In other words, we have the identity $\langle Av, w \rangle = \langle v, Bw \rangle$ for all v and w in V. It is now an easy exercise to check that B is a *linear* map from the complex vector space V to itself. This shows that any linear transformation on a finite-dimensional complex inner-product space has an adjoint. As above, we can easily show that if M is the matrix associated with A in an orthonormal basis, then its adjoint is given by $\overline{M}^t$. For that reason we sometimes use the notation $M^* = \overline{M}^t$.

Note: The above statements converting from linear transformations to matrices are *only* valid when one uses an orthonormal basis.

Orthogonal complements and Projections

Given a subspace W of a complex inner-product space V, we can define its orthogonal complement

$$W^\perp = \{v \in V | \langle v, w \rangle = 0 \forall w \in W\}$$

as the collection of all vectors v whose inner-product with *any* vector w in W is 0.

We note that the pairing *restricted* to W makes *it* into a complex inner product space. Thus, if W is finite dimensional, then we can find an orthonormal basis $e_1, \dots, e_m$ of W.

Given any vector v in V, we can define Pv by the formula

$$Pv = \sum_{i=1}^{m} \langle v, e_i \rangle e_i$$

(Note the similarity with the formula used above for Gram-Schmidt orthogonalisation.) It is clear that $P : V \to V$ is a linear transformation.

We check that $Pe_i = e_i$ for all i and Pv lies in W for every v in V. It follows that $P^2 = P$. So P is an *idempotent* transformation $V \to V$. Now,

$$\langle Pv, w \rangle = \sum_{i=1}^{m} \langle v, e_i \rangle e_i, w \rangle = \sum_{i=1}^{m} \overline{\langle w, e_i \rangle e_i, v \rangle} = \overline{\langle Pw, v \rangle} = \langle v, Pw \rangle$$

In other words, P is also self-adjoint, so it is a self-adjoint idempotent.

As we have seen in our study of idempotents P, we can write any vector v as $(1 - P)v + Pv$, in other words V is decomposed in the image of P and the image of $(1 - P)$. Further, if P is self-adjoint, then

$$\langle (1 - P)v, Pw \rangle = \langle P(1 - P)v, w \rangle = \langle 0, w \rangle = 0$$

Thus, if W is the image of P then the image of $1 - P$ is contained in $W^{\perp}$. Thus, V is the *orthogonal* sum of W and $W^{\perp}$.

In summary, a self-adjoint idempotent P (also called an orthogonal projection) leads to an orthogonal decomposition of the vector space into the image of P and the image of $1 - P$. If W is a finite dimensional subspace of V, then there is an orthogonal idempotent P whose image is W.

Eigenvector for an operator

Given a linear transformation $A : V \to V$ on a finite dimensional complex vector space, we can choose a basis of V and represent it as a matrix which we also denote by A. The polynomial equation $\det(A - T \cdot 1) = 0$ has a solution $T = \lambda$ over the field of complex numbers (by an application of the fundamental theorem of algebra). It follows that $\det(A - \lambda \cdot 1) = 0$. Thus, by the usual procedure for solving a finite system of linear equations, we can find a non-zero vector v so that $(A - \lambda \cdot 1)v = 0$; equivalently $Av = \lambda v$. Thus, we have proved that a linear transformation $A : V \to V$ on a finite dimensional complex vector space has an eigenvector.

Exercise: Consider the vector space V consisting of all polynomials in z^{-1} which can be considered as functions on the $\mathbb{C} - \{0\}$. The operator $D : V \to V$ sends each such polynomial function to its derivative. In other words $z^{-k} \mapsto -kz^{-k-1}$. Check that there is no eigenvector.

Thus, the finite dimensionality of V is important. For infinite dimensional spaces we will have to work harder to find eigenvectors!

Self-Adjoint Operators

Given a self-adjoint operator $A : V \to V$ on a complex inner-product space and an eigenvector $v \neq 0$, so that $Av = \lambda v$ for eigenvalue λ. We then have

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle$$

Since $v \neq 0$, we have $\langle v, v\rangle \neq 0$ so we get $\lambda = \overline{\lambda}$. In other words, we have shown that *any* eigenvalue of a self-adjoint operator is real.

If v is an eigenvector of A, let us consider the space

$$v^\perp = \{w \in V | \langle v, w\rangle = 0\}$$

We check that if w is in $v^\perp$, then

$$\langle v, Aw\rangle = \langle Av, w\rangle = \lambda\langle v, w\rangle = 0$$

Hence, A sends $v^\perp$ to itself. It is obvious that $v^\perp$ is a complex inner-product space and that the restriction of A to $v^\perp$ is self-adjoint.

In particular, we see by induction on dimension that given a self-adjoint operator A on a finite-dimensional complex inner product space V, there is an orthogonal (by scaling we can make it orthonormal) basis of V consisting of eigenvectors of A. Moreover, the eigenvalues are real numbers.

Unitary Operators

Given a unitary operator $U : V \to V$ on a complex inner-product space and an eigen vector $v \neq 0$, so that $Uv = \lambda v$ for eigenvalue λ. We then have

$$\langle v, v\rangle = \langle Uv, Uv\rangle = \langle \lambda v, \lambda v\rangle = |\lambda|^2\langle v, v\rangle$$

Hence, we see that $|\lambda|^2 = 1$. In other words, λ lies on the unit circle S^1 in the complex plane $\mathbb{C}$. Thus *any* eigenvalue of a unitary matrix has absolute value 1.

If v is an eigenvector of A, let us consider the space

$$v^\perp = \{w \in V | \langle v, w\rangle = 0\}$$

We check that if w is in $v^\perp$, then

$$\lambda\langle v, Uw\rangle = \langle Uv, Uw\rangle = \langle v, w\rangle = 0$$

Since $\lambda \neq 0$ (it has absolute value 1), it follows that Uw also lies in $v^\perp$. In other words, U takes $v^\perp$ to itself. It is obvious, as before, that $v^\perp$ is a complex inner-product space and that the restriction of U to $v^\perp$ is a unitary transformation.

In particular, we see by induction on dimension that give a self-adjoint operator U on a finite-dimensional complex inner product space V, there is an orthonormal basis of V consisting of eigenvectors of A. Moreover, the eigenvalues are of absolute value 1.

The above results can be extended to normal operators. We will show that a linear transformation of a finite dimensional inner product space to itself is normal if and only if there is an orthonormal basis consisting of eigenvectors of the transformation.