# Complex Numbers and Quaternions

We start by recalling the familiar properties of complex numbers.

## Complex Numbers

There is a product structure on 2-vectors defined by

$$(v_1, v_2) \odot (w_1, w_2) = (v_1 w_1 - v_2 w_2, v_1 w_2 + v_2 w_1)$$

This makes the vector space of 2-vectors over real numbers into a ring which is isomorphic to the ring $\mathbb{C}$ of complex numbers. The vector $(1, 0)$ plays the role of identity. We identify the real number $a$ with the element $(a, 0)$. The vector $i = (0, 1)$ has the property that $i \odot i = (-1, 0) = -1$. Moreover, for any complex number $v = (v_1, v_2)$ we define its *conjugate* as $\overline{v} = (v_1, -v_2)$ and its *norm*

$$\mathrm{Nm}(v) = v \odot \overline{v} = v_1^2 + v_2^2$$

For a non-zero vector $v$, $\mathrm{Nm}(v) \neq 0$, thus $\overline{v}/\mathrm{Nm}(v)$ is a *multiplicative inverse* of $v$. Thus, non-zero complex numbers form a group denoted by $\mathbb{C}^*$. Moreover, we have $\overline{v \odot w} = \overline{w} \odot \overline{v}$, and so

$$\mathrm{Nm}(v \odot w) = v \odot w \odot \overline{w} \odot \overline{v} = \mathrm{Nm}(v)\mathrm{Nm}(w)$$

(Actually $v \odot w = w \odot v$!)

Thus complex numbers of norm 1 form a subgroup sometimes denoted by $S^1$ since it is also the unit circle in the plane. We note that for a complex number of norm 1, we have $\overline{v} = v^{-1}$ is the multiplicative inverse.

The map $L_v : w \mapsto v \odot w$ is a R linear map from the 2-dimensional plane to itself. In the standard basis it is given by

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto L_v(w) = \begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

It follows that if $\mathrm{Nm}(v) = 1$, then $L_v$ is a plane rotation. This identifies plane rotations with group of unit complex numbers $S_1$. More generally, writing $v = \mathrm{Nm}(v)(v/\mathrm{Nm}(v))$ for a non-zero complex numbers identifies multiplication by a complex number with a rotation followed by a scaling (which is the same in all directions). In particular, this shows that multiplication by complex numbers preserves angles; in other words, this is *conformal*.

With his invention of quaternions, Hamilton tried to do something similar for rotations in 3 dimensions (and, as it turns out, in 4 dimensions as well).

## Quaternions

We use the symbol $(v, w)$ to denote the "dot-product" of the 3-vectors $v$ and $w$ and $v \times w$ to denote the "cross-product" of the 3-vectors $v$ and $w$. We then define the product

$$(a, v) \odot (b, w) = (ab - (v, w), aw + bv + v \times w)$$

Here $a$ and $b$ are real numbers and $v$ and $w$ are 3-vectors. One checks that this makes the 4-dimensional vector space consisting of pairs of the form $(a, v)$ into a ring with $(1, 0)$ playing the role of multiplicative identity. We use $\mathbb{H}$ to denote this ring and call it the ring of Quaternions. We identify elements of the form $(a, 0)$ with the corresponding real number $a$ and note that they commute with all other elements of $\mathbb{H}$.

For a quaternion $q = (a, v)$, we define its conjugate $\overline{q} = (a, -v)$. We also define its norm

$$\mathrm{Nm}(q) = q \odot \overline{q} = \big(a^2 + (v, v), 0\big) = a^2 + (v, v)$$

which is a real number. Now, if $q$ is a non-zero quaternion, then $\mathrm{Nm}(q)$ is non-zero and so $(1/\mathrm{Nm}(q))\overline{q}$ is a multiplicative inverse of $q$. Thus, non-zero quaternions form a group which we denote by $\mathbb{H}^*$.

Moreover, we have

$$\begin{aligned}\overline{(b, w)} \odot \overline{(a, v)} = (b, -w) \odot (a, -v) = \\ (ba - (w, v), -bv - aw + w \times v) = \\ (ab - (v, w), -aw - bv + v \times w) = \overline{(a, v) \odot (b, w)}\end{aligned}$$

Hence, for a pair $p$ and $q$ of quaternions, we have

$$\mathrm{Nm}(p \odot q) = p \odot q \odot \overline{q} \odot \overline{p} = \mathrm{Nm}(q) p \odot \overline{p} = \mathrm{Nm}(p)\mathrm{Nm}(q)$$

where we used the fact that $\mathrm{Nm}(q)$ is a real number which commutes with $p$.

Thus, quaternions of norm 1 form a subgroup which we can denote by $S^3$ since it consists of the vectors of unit length in 4 dimensional space. We note that for a quaternion of norm 1, we have $\overline{q} = q^{-1}$ is the multiplicative inverse.

So far, we see that the description of quaternions is very similar to the description of complex numbers. We now turn to the examination of the relation between quaternions and rotations.

## Rotations

First of all, let us note that $\mathrm{Nm}((a, v)) = a^2 + (v, v) = ((a, v), (a, v))$ which is the usual quadratic form on 4 dimensional space. If $q_1$ and $q_2$ are quaternions of norm 1, then, as seen above

$$\mathrm{Nm}(q_1 \odot (b,w) \odot \overline{q_2}) = \mathrm{Nm}((b,w))$$

In other words, the linear transformation $(b,w) \mapsto q_1 \odot (b,w) \odot \overline{q_2}$ is an orthogonal transformation $O(q1, q_2)$.

**Exercise**: Check that the map $(q_1, q_2) \mapsto O(q_1, q_2)$ is a group homomorphism from $S^3 \times S^3$ to the group $SO(4)$ of $4 \times 4$ orthogonal transformations of determinant 1.

One can show that this homomorphism is onto and has kernel $\{(1,1), (-1,-1)\}$.

Let us define the *trace* of a quaternion $q$ as $q + \overline{q}$ and denote it it as $\mathrm{Tr}(q)$. We note that $\mathrm{Tr}((a.v)) = 2a$. Secondly, we note that if $q$ is a unit quaternion, then

$$\begin{aligned}\mathrm{Tr}(q \odot (b,w) \odot \overline{q}) = q \odot (b,w) \odot \overline{q} + \overline{q \odot (b,w) \odot \overline{q}} = \\ q \odot (b,w) \odot \overline{q} + \overline{\overline{q}} \odot \overline{(b,w)} \odot \overline{q} = \\ q \odot (b,w)\overline{q} + q \odot \overline{(b,w)}\overline{q} = q\mathrm{Tr}((b,w))\overline{q} = 2b\end{aligned}$$

It follows that $(b,w) \mapsto q(b,w)\overline{q}$ *preserves* the trace of $(b,w)$. In particular, it takes the 3-dimensional subspace of quaternions of the form $(0,w)$ to itself by an orthogonal transformation. The precise expression if $q = (a,v)$ with $a^2 + (v,v) = 1$ is given below

$$(a,v) \odot (0,w) \odot (a,-v) = (0, (v,w)v + a^2 w + 2av \times w + v \times v \times w)$$

Thus we get an orthogonal transformation

$$O(a,v) : w \mapsto (v,w)v + a^2 w + 2av \times w + v \times v \times w$$

Note that we can write $(a,v) = (a, bu)$ where $(u,u) = 1$ and $a^2 + b^2 = 1$, which can used to simplify and explicate the above expression. Secondly, we note that

**Exercise**: Show that $v \times w \times v = (v,v)\left(w - \frac{(v,w)}{(v,v)}v\right)$.

Using the above exercise, one can prove

**Exercise**: Show that $O(a,v)$ is a rotation in the plane perpendicular to $v$ by an angle that is determined by $a$ and $(v,v)$.

**Exercise**: Show that $(a,v) \mapsto O(a,v)$ gives a group homomorphism from $S^3$ to the group $SO(3)$ of 3 dimensional rotations.

One can show that this homomorphism is onto and has kernel $\{\pm 1\}$.

## Special Unitary Matrices

We use the notation $\hat{i} = (0, e_1)$, $\hat{j} = (0, e_2)$ and $\hat{k} = (0, e_3)$ and note the identities

$$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = \hat{i}\hat{j}\hat{k} = -1$$

In other words, we can write every quaternion $q$ in the form

$$q = (q_0, (q_1, q_2, q_3)) = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k} = (q_0 + q_1\hat{i}) + \hat{j}(q_2 - q_3\hat{i})$$

Hence, we can think of a quaternion $q$ as a pair of complex numbers $(q_0 + q_1\iota, q_2 - q_3\iota)$. Written this way, the product becomes

$$(z_1, z_2) \odot (w_1, w_2) = (z_1 w_1 - \overline{z_2} w_2, \overline{z_1} w_2 + z_2 w_1)$$

Thus, multiplication by the quaternion $(z_1, z_2)$ as given by

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto U(z)w = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

We note that $\mathrm{Nm}(w) = |w_1|^2 + |w_2|^2$ where $|w_i|^2 = \mathrm{Nm}(w_i)$ when $w_i$ is a complex number. When $z$ is a quaternion of norm 1, multiplication by $U(z)$ preserves this "Hermitian dot-product" (a term that we will introduce in the next section). Moreover, we see that $\det U(z) = |z_1|^2 + |z_2|^2 = 1$. Thus, we have identified unit quaternions with the group of $2 \times 2$ special unitary matrices which will introduced shortly. This group is denoted as $SU(2)$.

The matrices

$$U(\hat{i}) = \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix}; U(\hat{j}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; U(\hat{k}) = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}$$

are sometimes called Pauli matrices in physics literature since they appear in phyics literature for the first time in a paper of Pauli. (The same ideas in greater generality had appeared in the work of Clifford decades earlier!)