Orthogonal Matrices

1. What are some values of b and c that will make the following an orthogonal matrix?

$$G = \begin{pmatrix} 3/5 & b \\ c & 3/5 \end{pmatrix}$$

2. For the above matrix (after substituting b and c) calculate the matrix $H = (G-1)(G+1)^{-1}$. Check that it is skew-symmetric.

3. Let $v = e_1 + e_2 + e3$ be the column 3-vector all of whose entries are 1. Write down the matrix of the associated reflection R_v defined as the linear transformation

$$R_v(w) = w - 2\frac{w^t v}{v^t v} v$$

4. More generally, if v is *any* column vector, then show that the matrix for R_v is given by

$$R_v = 1 - \frac{2}{v \cdot v} v v^t$$

(Note that vv^t is a 3×3 matrix!)

5. Using the above formula, write $A = R_v R_u$ in a form where it is "obvious" that A is identity on the vectors perpendicular to both u and v. Calculate the 2×2 matrix for A on the plane spanned by u and v (assume that u and v are linearly independent). Is it an orthogonal matrix? If not, why not?

6. For the quaternion $q = (1, (1, 1, 1))/2$ carry out the following:
   (a) Check that the norm of q is 1.
   (b) Write down the 4×4 matrix A of the linear transformation $(b, w) \mapsto q \odot (b, w)$.
   (c) Write down the 3×3 matrix B of the linear transformation $(0, w) \mapsto q \odot (0, w) \odot \overline{q}$.
   (d) Are A and B orthogonal matrices?
   (e) Describe the matrix B as a rotation. What is the angle in terms of sine and cosine?
   (f) Can you describe the matrix A as a rotation (is there any fixed "axis")?
   (g) What is the canonical form of B over complex numbers?

7. Fix a 3-vector v. Write the matrix for the linear transformation $w \mapsto v \times w$.

8. Repeat the above calculations for a *general* unit quaternion $q = (a, v)$ with $a^2 + v \cdot v = 1$.