Symmetric Bilinear Forms and Quadratic forms

1. Write down the quadratic form associated with the symmetric bilinear form $A(v, w) = v^t Aw$, where A is the 3×3 matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2. Write down the symmetric 3×3 matrix A associated with the quadratic form $Q(x, y, z)$ given by

$$Q(x, y, z) = xy + yz + z^2$$

3. Use completion of the square to write each quadratic form below in diagonal form.
   (a) $x^2 + 2xy + 2xz + 2y^2 + 3yz + 3z^2$
   (b) $x^2 + 2xy + 2xz + y^2 + yz + 2z^2$

4. In each of the examples in the question above write the associated symmetric matrix A that gives the bilinear form. Moreover, write the change of co-ordinates matrix S so that $S^t AS$ is in diagonal form.

5. Find the rank and signature of the following symmetric matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

6. Let v_1, v_2, v_3 be the column vectors of the matrix below. Apply Gram-Schmidt orthogonalisation to find a basis of orthogonal vectors of the form v_1, $v_2 + av_1$, $v_3 + bv_1 + cv_2$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

7. Given a symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Write down the conditions on the variables a, b and c that will make this a positive definite matrix.

8. (Starred) Find a similar condition for the 6 entries of a symmetric 3×3 matrix.

9. Given a symmetric matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.
   (a) Find a vector v (with real entries) with $v \cdot v = 1$ for which the quadratic form takes a maximum value.

(b) Using the above vector write an orthonormal matrix S so that $S^{-1}AS$ is diagonal.

10. (Starred) Given a symmetric matrix A over $\mathbb{Q}$, show that its minimal polynomial has distinct roots.

11. (Starred) If a symmetric matrix A is also nilpotent, then show that A is the 0 matrix.