Solutions to Second Mid-Sem Exam

(3 marks) 1. Find a 2×2 matrix S for which $S \cdot A \cdot S^{-1}$ is in diagonal form where A is given by

$$A = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

Solution: The minimal polynomial of this matrix divides $(T-3)(T-2)$ and since this is not a constant matrix, this must be its minimal polynomial. So we need to find vectors v and w so that $A \cdot v = 3v$ and $A \cdot w = 2 \cdot w$. Writing this in matrix form gives

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \cdot v = 0 \text{ and } \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \cdot w = 0$$

We see that the solution is given by

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, the solution is

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

So $S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

(5 marks) 2. Perform row and column operations to bring the following matrix into normal form diagonal (d_1, d_2, d_3) where d_1 divides d_2 and d_2 divides d_3.

$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

Solution: We subtract 3 multiple of row 1 from rows 2 and 3.

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -6 & -6 \\ 0 & -6 & -6 \end{pmatrix}$$

We subtract 2 multiple of column 1 from columns 2 and 3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & -6 \\ 0 & -6 & -6 \end{pmatrix}$$

We subtract row 2 from row 3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

We subtract column 2 from column 3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is in normal form as required.

(6 marks) 3. Which pairs of matrices in the following list are equivalent under the equivalence given by applying row and column operations? Which pairs are not? (Write your answer in the form $X \sim Y$ or $X \not\sim Y$ for each pair (X, Y) from the list below.)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}; C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 12 \end{pmatrix}; D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Solution: Matrices C and D are in normal form and have different diagonal entries so they are different. Matrices A and B are not in normal form. If we convert A to normal form we will get D and if we convert B to normal form we will get C.

$$A \not\sim B; A \not\sim C; A \sim D$$

$$B \sim C; B \not\sim D; C \not\sim D$$

(6 marks) 4. Which pairs of modules over $\mathbb{Q}[T]$ in the following list are isomorphic and which are not? Organise your answers in the form $X \cong Y$ or $X \not\cong Y$ for each pair (X, Y).

In case of the isomorphism write the map, in the case of non-isomorphism provide an example of an element of X for which there is no similar element in Y.

$$A = \mathbb{Q}[T]/(T^2 - 1)^2 \text{ and } B = \mathbb{Q}[T]/(T - 1)^2 \times \mathbb{Q}[T]/(T + 1)^2$$
$$\text{and } C = \mathbb{Q}[T]/(T^2 - 1) \times \mathbb{Q}[T]/(T^2 - 1)$$

Solution: Since $(T-1)^2$ and $(T+1)^2$ are co-prime, the Chinese Remainder Theorem implies that A and B are isomorphic. On the other hand $(T^2 - 1)$ acts as 0 on C whereas it is non-zero in its action on A. Thus, $A \cong B$, $B \ncong C$ and $A \ncong C$.

To exhibit an explicit isomorphism, we note $P(T) \mapsto (P(T), P(T))$ is an isomorphism $A \to B$. (This is proved in the notes for a general ring).

To exhibit an element of A which has no similar element on C, we take 1 in A. When multiplied by $T^2 - 1$ its image is not 0. On the other hand *every* element of C becomes 0 when multiplied by $T^2 - 1$.