## Solutions to Assignment 8

1. Given a module $M$ over $\mathbb{Q}[T]$, we can also think of it as a module (vector space) over $\mathbb{Q}$ (with something extra!). Check that the endomorphism $m \mapsto T \cdot m$ on $M$ is a linear transformation of the vector space $M$ over $\mathbb{Q}$.

> **Solution:** Since $T \cdot a = a \cdot T$ in the ring $\mathbb{Q}[T]$, it follows that
>
> $$T \cdot (a \cdot m) = (T \cdot a) \cdot m = (a \cdot T) \cdot m = a \cdot (T \cdot m)$$
>
> This proves that multiplication by $T$ commutes with multiplication by rational numbers. This is what is needed to make this a linear transformation since the additive property is anyway ensured by the fact that multiplication by $T$ is a group homomorphism $M \to M$.

2. Given a vector space $V$ over $\mathbb{Q}$ and a linear transformation $A : V \to V$, we can define $f : \mathbb{Q}[T] \to \mathrm{End}(V)$ by $f(P(T))(v) = P(A)(v)$ where

$$P(A)(v) = a_0 \cdot v + a_1 \cdot A(v) + \cdots + a_n \cdot A^n(v) \text{ when } P(T) = a_0 + a_1T + \cdots + a_nT^n$$

Check that $f$ is a ring homomorphism.

> **Solution:** A homomorphism $\mathbb{Q}[T] \to \mathrm{End}(V)$ is determined by what it does to $T$ since the action of $\mathbb{Q}$ is already determined. We are given that $T$ acts on $V$ by the linear transformation $A : V \to V$ which (by definition of a linear transformation) commutes with multiplication by rational numbers.

3. Show that the only ideals in a field $F$ are $F$ and $\{0\}$. and that a field is a domain. Conclude that a field is a PID.

> **Solution:** Any non-zero element $a$ in $F$ is a unit so that $a \cdot F = F$. Since $F$ is a domain and every ideal is principal, it is a principal ideal doamin.

4. If $D$ is a matrix in normal form over a field $F$, show that the diagonal entries of $D$ must be of a certain number of non-zero entries followed by 0's.

> **Solution:** Since every non-zero element is a unit, the non-zero elements divide each other and 0.

5. Use the above exercise to show that any finitely generated vector space has a basis.

**Solution:** We know from the structure for modules over a PID that a finitely generated module over a PID has the form $R/d_1 \times \cdots \times R/d_n$ where $d_i$ are the principal divisors. As shown above these $d_i$ are either units or 0, it follows that the module is free.

6. Show that $\mathbb{Q}[T]/(P(T)\mathbb{Q}[T])$ is a vector space over $\mathbb{Q}$ with basis given by $1, T, \ldots, T^{d-1}$ where $d$ is the degree of $P$.

**Solution:** Suppose $P(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0$. Since $P(T)T^k = 0$ in $\mathbb{Q}[T]/P(T)$ for all $k \geq 0$, we get the identities

$$T^{d+k} = -a_0 T^k - a_1 T^{k+1} - \cdots - a_{d-1} T^{k+d-1}$$

This means that all powers of $T$ higher than $T^{d=1}$ can be written in terms of $1, T, \ldots, T^{d-1}$. This shows that these elements generate the vector space. Since there is no polynomial of degree less than $d$ in $P(T) \cdot \mathbb{Q}[T]$, it follows that these elements are linearly independent as well.

7. In the above basis, check the matrix of the operation multiplication by $T$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -a_0 & -a_1 & a_2 & \cdots & -a_{d-1} \end{pmatrix}$$

where $P(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0$.

**Solution:** We have the identities

$$\begin{aligned} T \cdot 1 &= 0 \cdot 1 + 1 \cdot T + 0 \cdot T^2 + \cdots + 0 \cdot T^{d-1} \\ T \cdot T &= 0 \cdot 1 + 0 \cdot T + 1 \cdot T^2 + \cdots + 0 \cdot T^{d-1} \\ T \cdot T^{d-1} &= -a_0 \cdot 1 - a_1 \cdot T^1 - \cdots - a_{d-1} T^{d-1} \end{aligned}$$

This gives the matrix expression.

8. Check that for any polynomial $Q(T)$, the operation multiplication by $Q(T)$ on $\mathbb{Q}[T]/P(T)$ in the basis $1, T, \ldots, T^{d-1}$ is given by the matrix $Q(A)$ (see exercise 2 to see how $Q(A)$ is defined.)

**Solution:** The action of $\mathbb{Q}[T]$ on this vector space is determined by the action of multiplication by $T$; this is given by by the matrix $A$ as seen above. The resulting action of a matrix $Q(T)$ is then given by $Q(A)$.

9. Check that $P(A) = 0$. (Hint: Use the previous exercise.)

**Solution:** This follows immediately from the previous exercise.

10. As $\mathbb{Q}[T]/P(T)$ is a module over $\mathbb{Q}[T]$, we have the ring homomorphism

$$\mathbb{Q}[T] \to \mathrm{End}(\mathbb{Q}[T]/P(T)$$

Check that the kernel of this ring homomorphism is precisely $P(T) \cdot \mathbb{Q}[T]$.

**Solution:** This too follows from the previous exercise.

11. Choose a square matrix $A$ of size 3 (or 4) and carry out the row and column reductions on $T - A$ to calculate the basis in which it has the block form. Using this calculate its minimal polynomial and characteristic polynomial.

**Solution:** One such example was done in the previous solution set.

12. Given a $4 \times 4$ matrix $A$. In the normal form of $T - A$ what are the possible degrees of the diagonal entries (assume that we write them so that $P_1(T)|P_2(T)|P_3(T)|P_4(T)$. Using this find the possible sizes of the block form of $A$.

**Solution:** Let the degree of $P_i(T)$ be $d_i$. We then have $\sum_i d_i = 4$ and $d_i \leq d_{i+1}$. Thus, the possiblities for $(d_1, d_2, d_3, d_4)$ are

$$(0, 0, 0, 4); (0, 0, 1, 3); (0, 0, 2, 2); (0, 1, 1, 2); (1, 1, 1, 1)$$

In order to understand the block description we can use the above except that we can ignore the degree 0 terms. So the block sizes possible are (a) a single block of size 4, (b) a block of size 1 and a block of size 3, (c) two blocks of size 2, (d) 2 blocks of size 1 and a block of size 2, (e) 4 blocks of size 1.