Solutions to Assignment 6

1. (Starred) Show that for *any* non-zero 2×2 matrix A, the two-sided ideal generated by A in the ring $M_2(\mathbb{Q})$ of 2×2 matrices with rational entries, is the whole ring.

Solution: As seen before any non-zero matrix can be converted to diagonal form by row and column reduction operations. Thus, a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ lies in this two-sided ideal where either a or b is non-zero. From this it is not difficult to show that the matrix $E_{1,1}$ which has 0 except its $(1,1)$ entry is 1, is in this ideal. Now $E_{i,j} \cdot E_{j,k} = E_{i,k}$. So we can see that every element of the form $E_{i,j}$ is in the ideal. Now, every element is a linear combination of these matrices so the two-sided ideal is the whole ring.

2. Take any 3×3 matrix B with rational coefficients and consider the matrix $A = B - T \cdot 1$ where 1 denotes the identity matrix. Calculate the normal form of this matrix A. Repeat this a few times to ensure that you have understood all steps of the procedure. Try it with a 4×4 matrix for further practice.

Solution: We start with the matrix

$$B = \begin{pmatrix} 2 & 0 & -1 \\ 2 & 0 & -\frac{1}{2} \\ -2 & 1 & 0 \end{pmatrix}$$

This gives

$$A = \begin{pmatrix} 2-x & 0 & -1 \\ 2 & -x & -\frac{1}{2} \\ -2 & 1 & -x \end{pmatrix}$$

We use the entry 1 at $(3,2)$ as a pivot.

We add the x multiple of row 3 to row 2. Then we add the 2 multiple of column 2 to column 1 and the x multiple of column 2 to column 3.

$$A_1 = \begin{pmatrix} -x+2 & 0 & -1 \\ -2x+2 & 0 & -x^2-\frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

We now use the entry -1 at $(1,3)$ as the pivot.

We add the $-x^2 - \frac{1}{2}$ multiple of row 1 to row 2 and then we add the $-x+2$ multiple of column 3 to column 1.

$$A_3 = \begin{pmatrix} 0 & 0 & -1 \\ \frac{1}{2}(2\,x^2+1)(x-2) - 2\,x + 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Now interchanging column 3 and column 1, followed by the interchange of row 3 and row 1 with a change of sign once each time gives us a normal form.

$$A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2}(2\,x^2+1)(x-2)+2x-2 \end{pmatrix}$$

3. Check that $\mathbb{Z}/p$ is a field if and only if p is a prime number.

Solution: If p is not *prime* then there are positive integers a and b less than p so that $a \cdot b = 0$. This means that $a \cdot b = 0$ in $\mathbb{Z}/p$; thus, the latter is *not* a field as it has zero divisors.

If p is a *prime*, then every positive integer a which is less than it is co-prime to it. Thus we have x and y integers so that $a \cdot x + p \cdot y = 1$. It follows that $a \cdot x = 1$ in $\mathbb{Z}/p$. Hence, every non-zero element of the latter is a unit. Since it is commutative, it is a field.