Solutions to Assignment 5

1. Given an onto homomorphism $\mathbb{Z}^r \to M$, show that there are elements $a_1,\dots,a_r$ so that every element of M can be written as an additive combination of the elements a_i.

> **Solution:** Let e_i be the element of $\mathbb{Z}^r$ of the form $(0,\dots,1,\dots,0)$ where all entries are 0 except for a 1 in the i-th place. We take a_i to be the image of e_i under the homomorphism.
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> Since the homomorphism is onto, any element in m is the image of some element $(b_1,\dots,b_r)$ of $\mathbb{Z}^r$. This element can also be written as $\sum_{i=1}^r b_i \cdot e_i$, hence its image in M is $\sum_{i=1}^r b_i \cdot a_i$ (by the additivity of the homomorphism). Hence $m = \sum_{i=1}^r b_i \cdot a_i$.

2. Given an abelian group M and an idempotent p in $\mathrm{End}(M)$. Let $N = \ker(p) = \{a \in M : p(a) = 0\}$ be the kernel of p and $L = p(M)$ be the image of p. We have a natural group homomorphism $N \times L \to M$ given by $(n,l) \mapsto n + l$. Given any a in M we can put $l = p(a)$ and $n = a - p(a)$.

   (a) Check that $p(n) = 0$. Moreover, check that if $p(a)$ is in N, then $p(a) = 0$ so that $N \cap L = 0$.

   > **Solution:** We have
   >
   > $$p(n) = p(a - p(a)) = p(a) - p(p(a)) = p(a) - p^2(a) = p(a) - p(a) = 0$$

   (b) Conclude that $N \times L \to M$ is an isomorphism (i.e. it is one-to-one and onto).

   > **Solution:** Clearly $a = (a - p(a)) + p(a) = n + l$, so that map is onto. If $a = n + l = 0$, then $l = p(a) = p(0) = 0$ and then $n = a - p(a) = 0 - 0 = 0$. This shows that the map is one-to-one.

3. Suppose that we have a group homomorphism $f : M \to \mathbb{Z}^r$ for some r and that this map is *onto*. For each i between 1 and r we have the element e_i of $\mathbb{Z}^r$ which has 1 in the i-th place and 0 elsewhere. Since f is onto, there is an element a_i of M such that $f(a_i) = e_i$. We define a homomorphism $g : \mathbb{Z}^r \to M$ so that $g(e_i) = a_i$.

   (a) Show that $f \circ g$ is the identity endomorphism of $\mathbb{Z}^r$.

   > **Solution:** First of all, we note that $(f \circ g)(e_i) = f(g(e_i)) = f(a_i) = e_i$ by the choice of a_i. Secondly, we note (as above) that every element $(b_1,\dots,b_r)$ of $\mathbb{Z}^r$ is expressible as $\sum_{i=1}^r b_i \cdot e_i$. Hence, its image under $f \circ g$ is itself.

(b) Show that g is one-to-one so that $g(\mathbb{Z}^r)$ can be thought of as a copy of $\mathbb{Z}^r$ inside M.

Solution: If $g(b_1, \ldots, b_r) = 0$, then $f(g(b_1, \ldots, b_r)) = (0, \ldots, 0)$. It follows that

$$(0, \ldots, 0) = (f \circ g)(b_1, \ldots, b_r)) = (b_1, \ldots, b_r)$$

(c) Show that $p = g \circ f$ is an idempotent endomorphism of M.

Solution: We check $p \circ p = g \circ f \circ g \circ f = g \circ u \circ f$ where $u = f \circ g$ is the identity map on $\mathbb{Z}^r$. It follows that the $g \circ u \circ f = g \circ f = p$, so that $p \circ p = p$.

(d) Show that M is isomorphic to $\ker(f) \times \mathbb{Z}^r$.

Solution: We have produced an idempotent p on M. Hence M is isomorphic to the sum of the kernel of p and the image of p as shown above. The image of p is the same as the image of g since f is onto; since g is one-to-one, the image of g can be identified with $\mathbb{Z}^r$. On the other hand the kernel of p is the same as the kernel of f since g is one-to-one.

4. Select a 4×4 integer matrix A and reduce it to normal form using row and column reductions. Do it a few times with different matrices to make sure that all the steps outlined in the notes are used! Increase the size to 5×5 for extra practice.

Solution: We start with the matrix

$$A = \begin{pmatrix} 11 & 0 & 7 & 1 \\ -16 & 3 & 1 & 1 \\ 0 & 3 & -1 & -5 \\ -4 & 0 & 0 & 2 \end{pmatrix}$$

We start with the pivot as the 1 in the $(1, 4)$ entry in the top right corner.

We add the -1 (respectively 5 and -2) of row 1 to row 2 (respectively 3 and 4) to get:

$$A_1 = \begin{pmatrix} 11 & 0 & 7 & 1 \\ -27 & 3 & -6 & 0 \\ 55 & 3 & 34 & 0 \\ -26 & 0 & -14 & 0 \end{pmatrix}$$

We add the -11 (respectively -7) of column 4 to row 1 (respectively 3) to get:

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -27 & 3 & -6 & 0 \\ 55 & 3 & 34 & 0 \\ -26 & 0 & -14 & 0 \end{pmatrix}$$

We next have the pivot as the 3 in the $(2,2)$ entry.

We add the -1 multiple of row 2 to row 3, the 9 mutliple of column 2 to column 1 and the 2 multiple of column 2 to column 3.

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 82 & 0 & 40 & 0 \\ -26 & 0 & -14 & 0 \end{pmatrix}$$

We next have the pivot as the -14 in the $(4,3)$ entry.

We add the -2 multiple of column 3 to column 1.

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 40 & 0 \\ 2 & 0 & -14 & 0 \end{pmatrix}$$

We next have the pivot as the 2 in the $(3,1)$ entry.

We add the -1 multiple of row 3 to row 4 and the -20 multiple of column 1 to column 3.

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -54 & 0 \end{pmatrix}$$

Interchanging column 1 with column 4 and then column 3 with column 4 and switching one sign each time.

$$A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -54 \end{pmatrix}$$

Now it is in diagonal form, but not in normal form since 3 does not divide 2. So we add row 3 to row 2 and subtract column 3 from column 2.

$$A_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -54 \end{pmatrix}$$

We now use the 1 in the $(2,2)$ entry as the pivot.

We add the 2 multiple of row 2 to row 3 and then the -2 multiple of column 2 to column 3.

$$A_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -54 \end{pmatrix}$$

Since 6 divides -54, this is in normal form.