Modules over $\mathbb{Q}[T]$

1. Given a module M over $\mathbb{Q}[T]$, we can also think of it as a module (vector space) over $\mathbb{Q}$ (with something extra!). Check that the endomorphism $m \mapsto T \cdot m$ on M is a linear transformation of the vector space M over $\mathbb{Q}$.

2. Given a vector space V over $\mathbb{Q}$ and a linear transformation $A : V \to V$, we can define $f : \mathbb{Q}[T] \to \mathrm{End}(V)$ by $f(P(T))(v) = P(A)(v)$ where

   $$P(A)(v) = a_0 \cdot v + a_1 \cdot A(v) + \cdots + a_n \cdot A^n(v) \text{ when } P(T) = a_0 + a_1 T + \cdots + a_n T^n$$

   Check that f is a ring homomorphism.

3. Show that the only ideals in a field F are F and $\{0\}$. and that a field is a domain. Conclude that a field is a PID.

4. If D is a matrix in normal form over a field F, show that the diagonal entries of D must be of a certain number of non-zero entries followed by 0's.

5. Use the above exercise to show that any finitely generated vector space has a basis.

6. Show that $\mathbb{Q}[T]/(P(T)\mathbb{Q}[T])$ is a vector space over $\mathbb{Q}$ with basis given by $1, T, \ldots, T^{d-1}$ where d is the degree of P.

7. In the above basis, check the matrix of the operation multiplication by T is given by

   $$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -a_0 & -a_1 & a_2 & \cdots & -a_{d-1} \end{pmatrix}$$

   where $P(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0$.

8. Check that for any polynomial $Q(T)$, the operation multiplication by $Q(T)$ on $\mathbb{Q}[T]/P(T)$ in the basis $1, T, \ldots, T^{d-1}$ is given by the matrix $Q(A)$ (see exercise 2 to see how $Q(A)$ is defined.)

9. Check that $P(A) = 0$. (Hint: Use the previous exercise.)

10. As $\mathbb{Q}[T]/P(T)$ is a module over $\mathbb{Q}[T]$, we have the ring homomorphism

    $$\mathbb{Q}[T] \to \mathrm{End}(\mathbb{Q}[T]/P(T)$$

    Check that the kernel of this ring homomorphism is precisely $P(T) \cdot \mathbb{Q}[T]$.

11. Choose a square matrix A of size 3 (or 4) and carry out the row and column reductions on $T - A$ to calculate the basis in which it has the block form. Using this calculate its minimal polynomial and characteristic polynomial.

12. Given a 4×4 matrix A. In the normal form of $T - A$ what are the possible degrees of the diagonal entries (assume that we write them so that $P_1(T)|P_2(T)|P_3(T)|P_4(T)$. Using this find the possible sizes of the block form of A.