Matrices Over Polynomial Rings

1. (Starred) Show that for *any* non-zero 2×2 matrix A, the two-sided ideal generated by A in the ring $M_2(\mathbb{Q})$ of 2×2 matrices with rational entries, is the whole ring.

2. Take any 3×3 matrix B with rational coefficients and consider the matrix $A = B - T \cdot 1$ where 1 denotes the identity matrix. Calculate the normal form of this matrix A. Repeat this a few times to ensure that you have understood all steps of the procedure. Try it with a 4×4 matrix for further practice.

3. Check that $\mathbb{Z}/p$ is a field if and only if p is a prime number.