Matrices Over Integers

1. Given an onto homomorphism $\mathbb{Z}^r \to M$, show that there are elements $a_1, \ldots, a_r$ so that every element of M can be written as an additive combination of the elements a_i.

2. Given an abelian group M and an idempotent p in $\mathrm{End}(M)$. Let $N = \ker(p) = \{a \in M : p(a) = 0\}$ be the kernel of p and $L = p(M)$ be the image of p. We have a natural group homomorphism $N \times L \to M$ given by $(n, l) \mapsto n + l$. Given any a in M we can put $l = p(a)$ and $n = a - p(a)$.
   (a) Check that $p(n) = 0$. Moreover, check that if $p(a)$ is in N, then $p(a) = 0$ so that $N \cap L = 0$.
   (b) Conclude that $N \times L \to M$ is an isomorphism (i.e. it is one-to-one and onto).

3. Suppose that we have a group homomorphism $f : M \to \mathbb{Z}^r$ for some r and that this map is *onto*. For each i between 1 and r we have the element e_i of $\mathbb{Z}^r$ which has 1 in the i-th place and 0 elsewhere. Since f is onto, there is an element a_i of M such that $f(a_i) = e_i$. We define a homomorphism $g : \mathbb{Z}^r \to M$ so that $g(e_i) = a_i$.
   (a) Show that $f \circ g$ is the identity endomorphism of $\mathbb{Z}^r$.
   (b) Show that g is one-to-one so that $g(\mathbb{Z}^r)$ can be thought of as a copy of $\mathbb{Z}^r$ inside M.
   (c) Show that $p = g \circ f$ is an idempotent endomorphism of M.
   (d) Show that M is isomorphic to $\ker(f) \times \mathbb{Z}^r$.

4. Select a 4×4 integer matrix A and reduce it to normal form using row and column reductions. Do it a few times with different matrices to make sure that all the steps outlined in the notes are used! Increase the size to 5×5 for extra practice.