Solutions to First Mid-Sem Exam

1. Give *one* example of the following types of elements in the following rings, *or* indicate that there is no such element. (0 denotes the additive identity of the ring, 1 denotes the multiplicative identity and -1 denotes the additive inverse of 1.)

(1 mark) (a) An idempotent element in $\mathbb{Z}/42$ which is different from 0 or 1.

Solution: The element 7 has the property that $7^2 = 7$ in $\mathbb{Z}/42$.

(1 mark) (b) An idempotent element in $M_2(\mathbb{Z})$ which is different from 0 or 1.

Solution: The element $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has the property that $A^2 = A$ in $M_2(\mathbb{Z})$.

(1 mark) (c) An idempotent element in $\mathbb{Z}$ which is different from 0 or 1.

Solution: There is no such element.

(1 mark) (d) A unit in $\mathbb{Z}/42$ which is different from 1.

Solution: The element 5 has the property that $5 \cdot 17 = 1 = 17 \cdot 5$ in $\mathbb{Z}/42$.

(1 mark) (e) A unit in $M_2(\mathbb{Z})$ which is different from 1 or -1.

Solution: The element $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has the property that $A^2 = 1$ in $M_2(\mathbb{Z})$.

(1 mark) (f) A unit in $\mathbb{Z}$ which is different from 1 or -1.

Solution: There is no such element.

(1 mark) (g) A nilpotent element in $\mathbb{Z}/42$ which is different from 0.

Solution: There is no such element.

(1 mark) (h) A nilpotent element in $M_2(\mathbb{Z})$ which is different from 0.

Solution: The element $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has the property that $A^2 = 0$ in $M_2(\mathbb{Z})$.

(1 mark) (i) A nilpotent element in $\mathbb{Z}$ which is different from 0.

Solution: There is no such element.

(1 mark) (j) A zero-divisor in $\mathbb{Z}/42$ which is different from 0.

Solution: The element 2 has the property that $2 \cdot 21 = 0$ in $\mathbb{Z}/42$ and 21 is not zero in $\mathbb{Z}/42$.

(1 mark) 2. Give an example of a ring where 1 is its own additive inverse *or* indicate that there is no such ring.

Solution: The ring $\mathbb{Z}/2$.

3. We note that $S = \mathbb{Z}/2 \times \mathbb{Z}/3$ is a ring with component-wise addition and multiplication. Consider the element $a = (1, 2)$ in S.

(2 marks) (a) Consider the map $k \mapsto k \cdot a$ from $\mathbb{Z}/6$ to S. Is this a group isomorphism? Is this a ring isomorphism?

Solution: This is a group isomorphism since a is an element of order 6 and S has 6 elements.

This is not a ring isomorphism since 1 maps to $(1, 2)$ which is not the multiplicative identity of S.

(1 mark) (b) Are the rings S and $\mathbb{Z}/6$ isomorphic?

Solution: The element $b = (1, 1)$ is the multiplicative identity of S and it is also an element of order 6. Hence, the map $k \mapsto k \cdot b$ is a ring isomorphism.

(3 marks) 4. In a commutative ring R we are given an element a that satisfies $a^2 + 1 = 0$ and an element b so that $b^2 + b^2 = 1$, show that if $c = (1 + a) \cdot b$, then $c^2 = a$.

Solution:

$$
\begin{aligned}
c^2 &= ((1+a)\cdot b)\cdot((1+a)\cdot a) \\
&= (1+a)^2 \cdot b^2 \\
&\text{using associative and commutative laws} \\
&\text{for multiplication} \\
&= (1+a+a+a^2))\cdot b^2 \\
&\text{using distributive law and associative law} \\
&\text{for addition and multiplicative identity} \\
&= (a+a)\cdot b^2 \\
&\text{using commutative law for addition} \\
&\text{and } 1+a^2=0 \text{ and additive identity} \\
&= a\cdot b^2 + a\cdot b^2 \\
&\text{using distributive law} \\
&= a\cdot(b^2+b^2) \\
&\text{using distributive law} \\
&= a\cdot 1 \\
&\text{using the equation } b^2+b^2=1 \\
&= a \\
&\text{using the multiplicative identity}
\end{aligned}
$$

(3 marks) 5. In a ring R, let 0 denote the additive identity and a denote an arbitrary element of R.

By *only* using the axioms for a ring show that $0\cdot a = 0$. In each step make it clear which axiom you are using.

Solution:

$$
\begin{aligned}
0 &= 0 + 0 \\
&\quad \text{using additive identity} \\
0 \cdot a &= (0 + 0) \cdot a \\
&\quad \text{multiplying both sides by } a \\
&= 0 \cdot a + 0 \cdot a \\
&\quad \text{using distributive law} \\
0 \cdot a + b &= (0 \cdot a + 0 \cdot a) + b \\
&\quad \text{adding the additive inverse } b \text{ of } 0 \cdot a \\
0 &= 0 \cdot a + (0 \cdot a + b) \\
&\quad \text{using the additive inverse propery} \\
&\quad \text{and the associative law} \\
0 &= 0 \cdot a \\
&\quad \text{using the additive inverse propery} \\
&\quad \text{and the additive identity}
\end{aligned}
$$