Abelian Groups — Part 1

1. Given that f and g are endomorphisms of an abelian group M. Define $h(a) = f(a)+g(a)$ for every a in M. Check that h is an endomorphism of the abelian group M.

2. Given that f and g are endomorphisms of an abelian group M. Define $h(a) = f(g(a))$ for every a in M. Check that h is an endomorphism of the abelian group M.

3. The $\underline{0}$ endomorphism of an abelian group M sends every element of M to 0. The $\underline{1}$ endomorphism of M is the identity map of M to itself. With the above definitions of addition and multiplications, check that $\text{End}(M)$ is a ring.

4. Consider the natural ring homomorphism $\eta : \mathbb{Z} \to \text{End}(M)$ given that the latter is a ring. Describe the element $\eta(3)$ and $\eta(-2)$. How do you prove that this description is correct?

5. Use the (left) distributive law in a ring R to show that $x \mapsto a \cdot x$ is an endomorphism of $(R, +)$.

6. Call the map in the previous exercise ℓ_a. Use associativity of multiplication in R to show that $\ell_a \circ \ell_b = \ell_{a \cdot b}$.

7. Use the right distributive law in R to show that $\ell_{a+b} = \ell_a + \ell_b$ where the right-hand side is addition of endomorphisms of $(R, +)$.

8. Use the additive identity law to show that ℓ_1 is the identity endomorphism of $(R, +)$.

9. If p is any integer and a an element of an abelian group M, then show that the order of $p \cdot a$ divides the order m of a.

10. In the above situation, if p and m have no common factor, then show that the order of $p \cdot a$ is m.

11. In the above situation, Show that the element $k \cdot a$ has order *exactly* m/k.

12. Combine the above two exercises to show that if the order of $p \cdot a$ is m/k where k the greatest common divisor of p and m.

13. Given two elements a and b in an abelian group M with orders m and n respectively. If m and n have no common factor then show that if some multiple $p \cdot a$ equals some multiple $q \cdot b$ then both of these are the 0 element of M. (Hint: The order of $p \cdot a$ is a divisor of m and that of $q \cdot b$ is a divisor of n.)

14. In the above situation, if m and n have no common factor then the order of $a + b$ is $m \cdot n$.

15. Given positive integers m and n, show that there is a divisor k of m and a divisor l of n so that:

    1. k and l have no common factor.

2. The least common multiple of m and n is $k \cdot l$.

(Hint: Let k be product of those prime powers dividing m that are the same as the prime powers dividing the least common multiple of m and n. Let l be the product of the remaining prime powers dividing the l.c.m. of m and n.)

16. With a, b in M as above and m, n, k and l positive integers as above, show that $(m/k) \cdot a + (n/l) \cdot b$ has order equal to the least common multiple of the m and n.

17. (Starred) Given a and b in M of order m and n respectively, show that the order of any element of the form $p \cdot a + q \cdot b$ divides the least common multiple of m and n.

18. Check that the order of every non-zero element of $\mathbb{Z}/3 \times \mathbb{Z}/3$ is 3.

19. Check that the order of the element $(1, 1)$ of $\mathbb{Z}/4 \times \mathbb{Z}/9$ is 36.

20. Find an element of order 6 in $\mathbb{Z}/4 \times \mathbb{Z}/9$.

21. If a is an element of an (abelian) group M, and N is a subgroup of G. Suppose $s \cdot a$ and $t \cdot a$ lie in N. Show that $p \cdot a$ lies in N where p is the greatest common divisor of s and t. (Hint: p can be written as an additive combination of s and u.)

22. Suppose that u divides n and s divides u. Now if k is a divisor of n so that $n/k = u/s$, show that s divides k.