Variables, Polynomials, Functions and Constants

1. Check that the only idempotents in $\mathbb{Z}$ are 0 and 1.

2. For what integers n can you find idempotents *different from* 0 and 1 in $\mathbb{Z}/n$?

3. Given any ring R we have a natural ring homomorphism $f : \mathbb{Z} \to R$. For any element a in R and any integer n, check that $f(n) \cdot a = a \cdot f(n)$.

4. Given an element a in a ring R consider the two "new" elements $b = 2 + 3 \cdot a$ and $c = a - 5 \cdot a^3$. Check that $b \cdot c$ has the form $n_0 + n_1 \cdot a + n_2 \cdot a^2 + n_3 \cdot a^3 + n_4 \cdot a^4$. How did you use the previous exercise in solving this one?

5. Write down the formulas for addition and multiplication of $p(T) = p_0 + p_1 T + \cdots + p_k T^k$ and $q(T) = q_0 + q_1 T + \cdots + q_l T^l$. Here k and l are non-negative integers and p_i's and q_j's are elements of a ring R.

6. (Starred) For a ring S and a fixed element s in S, define a map $D_s(a) = s \cdot a - a \cdot s$. This is *not* a ring homomorphism. However, check that $D_s(a + b) = D_s(a) + D_s(b)$ and (more importantly) $D_s(a \cdot b) = a \cdot D_s(b) + D_s(a) \cdot b$.

7. Suppose that R is commutative and that S is an R-algebra. Show that giving an element of S is the same as giving a homomorphism $R[T] \to S$ where the map is the natural one on R.

8. Suppose $a \cdot b \neq b \cdot a$ in R, then show that the map $R[T] \to R$ which sends T to a is *not* a homomorphism.

9. Check that point-wise addition and multiplication make $\mathrm{Map}(X, R)$ into a ring for any set X

10. For each element a in R we can consider the "constant" function $\underline{a}$ which sends every element of X to a. Show that this gives a ring homomorphism $R \to \mathrm{Map}(X, R)$.

11. Check that evaluation gives a ring homomorphism $R[T] \to \mathrm{Map}(R, R)$ when R is commutative.

12. (Starred) Does the above statement hold if R is not commutative? Give an example to justify your answer.

13. How many elements are there in the set $\mathrm{Map}(\mathbb{Z}/n, \mathbb{Z}/n)$?

14. For $n = 3, 4, 5, 6$, find an explicit polynomial $p(T)$ in $(\mathbb{Z}/n)[T]$ for which $e_p(k) = 0$ for *every* element k in $\mathbb{Z}/n$.

15. Find an explicit polynomial $p(T)$ in $(\mathbb{Z}/n)[T]$ for which $e_p(k) = 0$ for *every* element k in $\mathbb{Z}/n$.