Some more rings and their elements

1. Show that the formula for multiplication of matrices over a ring R follows the distributive and associative laws in R.

2. Using the associative law show that if $x^2 = r$ and $y^2 = s$ and $x \cdot y = -(y \cdot x)$, then $x \cdot y)^2 = -r \cdot s$.

3. Consider the ring $\mathbb{H}$ consisting of pairs $(a, \vec{v})$ where a is a real number and $\vec{v}$ is a vector in 3-dimensional space. Addition is carried out component-wise and multiplication is given by:
$$(a, \vec{v}) \cdot (b, \vec{w}) = (a \cdot b - \vec{v} \cdot \vec{w}, \vec{v} \times \vec{w})$$
where $\vec{v} \cdot \vec{w}$ is the usual dot-product and $\vec{v} \times \vec{w}$ is the usual cross-product.

   Check that $\mathbb{H}$ is a ring under these operations.

4. Check that the two ways of constructing $\mathbb{H}$ via matrices and as given above result in the same ring via a natural correspondence.

5. (Starred) We take the set S to consist of n-tuples of elements of R and define the operations on S via the rules:

   - addition is defined component-wise.
   - for the tuples $e_i = (0, \dots, 1, \dots, 0)$ (where 1 is in the i-th place) we define mutliplications $e_i \cdot e_j$ as linear combinations (using elements $c_{i,j,k}$ in R):

   – Check that the associative law for multiplication requires some identities to hold in R for the elements $c_{i,j,k}$.

   Rather than check such identities each time, it is easier to see these examples as special cases of "matrices" as we shall do below. In that case, the associative law needs to be checked just once!

6. (Starred) Why do we define matrix multiplication the way we do?

7. Given a commutative ring R and an element r of R show that matrices of the form:
$$\begin{pmatrix} a & b \cdot r \\ b & a \end{pmatrix}$$
are closed under addition and mutliplication; here a and b denote elements of R.

8. Show that the the ring of complex numbers is the same as the collection of matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where a and b are real numbers.

9. Check that 2×2 matrices of the type $\begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix}$ with u and v in the field of complex numbers is closed under addition and multiplication.

10. Check that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is a nilpotent matrix.
11. What are the nilpotent elements in the ring $\mathbb{Z}/24$?
12. Check that an upper triangular matrix is nilpotent.
13. (Starred) Give an example of a matrix which is *not* upper or lower triangular and yet is nilpotent.
14. Check that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an idempotent matrix.
15. Are there any idempotent elements in $\mathbb{Z}/6$ other than 1?
16. If p is an idempotent element, then check that $1 - p$ is also an idempotent element.
17. If p is an idempotent element of a ring R, then check that the set $pRp = \{pap | a \in R\}$ is closed under addition and multiplication and that p acts as multiplicative identity on pRp.
18. Check that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a zero divisor.
19. What are the zero divisors in the ring $\mathbb{Z}/42$?
20. Give an example of a 2×2 matrix which is not nilpotent and not idempotent and yet is a zero divisor.
21. Find a condition under which an element k of $\mathbb{Z}/n$ is a zero divisor.
22. (Starred) Find the condition under which a 2×2 matrix over rational numbers is a zero divisor in this ring.
23. Check that the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is a unit in the ring of 2×2 matrices.
24. Check that 5 is a unit in the ring $\mathbb{Z}/42$.
25. Check that the product of units is also a unit.
26. Give a condition on an element k of $\mathbb{Z}/n$ so that it is a unit in this ring.
27. (Starred) Give a condition on 2×2 matrices over rational numbers so that it is a unit in this ring.
28. If u is a unit *and* is idempotent, then check that $u = 1$.

29. Can there be a unit which is also a zero divisor?

30. Is it possible for the sum of nilpotent elements to be a unit?

31. Is it possible for the sum of units to be nilpotent?

32. (Starred) Ask yourself other questions about other combinations of properties and come up with their answers!

33. Note that the only idempotents in $\mathbb{Z}$ are 0 and 1.

34. Show that the map that sends an element a or R to the $p \times p$ matrix $f(a)$ which has a on the diagonal and 0 everywhere else gives a ring homomorphism $f : R \to M_p(R)$.

35. Check that A given by
$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & \end{pmatrix}$$
satisfies the equation
$$X^n + a_{n-1}X^{n-1} + \dots + a_0 = 0$$

36. Check that $2 \cdot 2 + 1 = 0$ in the ring $\mathbb{Z}/5$.

37. Check that $2 \cdot 2 \cdot 2 - 1 = 0$ in the ring $\mathbb{Z}/7$.