Special types of elements in rings

Now that we have a number of different examples of rings, we can look for properties of elements.

Nilpotent Elements

An element of a ring R is called *nilpotent* if some power of it is 0. Note that 0 *is* a nilotent element!

Exercise: Check that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is a nilpotent matrix.

Exercise: What are the nilpotent elements in the ring $\mathbb{Z}/24$?

A (square) matrix is called *upper triangular* if all elements on and below the diagonal are 0.

Exercise: Check that an upper triangular matrix is nilpotent.

Of course, a similar statement holds for *lower triangular* matrices, which have all entries on or above the diagonal as zero.

Exercise: (Starred) Give an example of a matrix which is *not* upper or lower triangular and yet is nilpotent.

Idempotent Elements

An element p of a ring is called *idempotent* if $p^2 = p \cdot p = p$. Note that 1 is *always* an idempotent element!

Exercise: Check that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an idempotent matrix.

Exercise: Are there any idempotent elements in $\mathbb{Z}/6$ other than 1?

Exercise: If p is an idempotent element, then check that $1 - p$ is also an idempotent element.

Exercise: If p is an idempotent element of a ring R, then check that the set $pRp = \{pap | a \in R\}$ is closed under addition and multiplication and that p acts as multiplicative identity on pRp.

In this case pRp is a ring. However, it is *not* a sub-ring of R since the identity element is not the same.

Zero divisors

An element a of a ring R is called a *zero divisor* if there is a non-zero element b of R so that $a \cdot b = 0$ or $b \cdot a = 0$. Note that every nilpotent matrix *is* a zero divisor by this definition—what about 0?! Sometimes, it is useful to distinguish between left zero divisors and right zero divisors.

Exercise: Check that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a zero divisor.

Exercise: What are the zero divisors in the ring $\mathbb{Z}/42$?

Exercise: Give an example of a 2×2 matrix which is not nilpotent and not idempotent and yet is a zero divisor.

Exercise: Find a condition under which an element k of $\mathbb{Z}/n$ is a zero divisor.

Exercise: (Starred) Find the condition under which a 2×2 matrix over rational numbers is a zero divisor in this ring.

Units

An element u of a ring R is called a *unit* if there is an element v of R for which $u \cdot v = 1 = v \cdot u$. Note that 1 is automatically a unit! Sometimes it is useful to talk about left units and right units.

Exercise: Check that the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is a unit in the ring of 2×2 matrices.

Exercise: Check that 5 is a unit in the ring $\mathbb{Z}/42$.

Exercise: Check that the product of units is also a unit.

Exercise: Give a condition on an element k of $\mathbb{Z}/n$ so that it is a unit in this ring.

Exercise: (Starred) Give a condition on 2×2 matrices over rational numbers so that it is a unit in this ring.

Combinations of types

We can play with the inter-relations between the above conditions.

Exercise: If u is a unit *and* is idempotent, then check that $u = 1$.

Exercise: Can there be a unit which is also a zero divisor?

Exercise: Is it possible for the sum of nilpotent elements to be a unit?

Exercise: Is it possible for the sum of units to be nilpotent?

Exercise: (Starred) Ask yourself other questions about other combinations of properties and come up with their answers!