Rings, Ideals and Homomorphisms

1. Show that $0 = 1$ in a ring if and only if the ring consists of just one element 0 with $0 + 0 = 0$ and $0 \cdot 0 = 0$.

2. In a ring, check that $a.0 = 0 = 0.a$ for any element a of the ring.

3. Check that the axioms of a ring are satisfied by $\mathbb{Z}/n$. (Hint: One can always take remainder "at the end.")

4. Check that the program below calculates the greatest common divisor of a and b. (Hint: We only need to check that the greatest common divisor is *invariant* under the above substitutions.)

```
def gcd(a,b):
    a, b = abs(a), abs(b)
    if b > a:
        a, b = b, a
    while b != 0:
        a, b = b, a%b
    return a
```

5. Given three numbers a, b and c, we can calculate $d = gcd(gcd(a, b), c)$. Check that d is the greatest common divisor of a, b and c.

6. If the greatest common divisor of S is d then show that any multiple of d can be written as a *finite* additive combination of multiples of elements of S.

7. Consider the set R of real numbers of the form $a + b\sqrt{5}$ where a and b are *integers* with the usual operations of addition and multiplication of real numbers. Check that R as defined above is a ring.

8. Show that $(m\mathbb{Z}) \cdot (n\mathbb{Z}) = (mn) \cdot \mathbb{Z}$ and $(m\mathbb{Z}) + (n\mathbb{Z}) = \gcd(m, n)\mathbb{Z}$.

9. More generally, for any ring R and ideals I and J in R, show that $I \cdot J$ and $I + J$ are ideals in R.

10. Given a ring R, we can define a set map $r : \mathbb{Z} \to R$ by defining the image of 0 as 0 (in R), the image of a positive integer n is the sum of n copies of 1 (in R), the image of a negative integer $-n$ is the sum of n copies of -1 (in R).

    Check that the above map r has the property that $r(m+n) = r(m)+r(n)$ and $r(m\cdot n) = r(m) \cdot r(n)$.

11. If $f : R \to S$ is a homomorphism of rings then define the set I to consist of elements a such that $f(a) = 0$. Check that I is an ideal.

12. What are the elements a and a' of R such that $a + I = a' + I$?

13. Check that R/I with the operations $\oplus$ and $\odot$ as addition and multiplication forms a ring with $0 + I$ and $1 + I$ as additive and multiplicative identity respectively.

14. **Starred** Look for other examples of rings that you have already learned about so far.