Introduction

The course outline given in the Courses of Study is reproduced below.

MTH302: Integers, polynomials and matrices

Credits	4
Lectures	3
Tutorials	1
Labs	0

Course Outline

- Definitions and examples of Rings, integral domains, division rings and fields.
- Ideals, maximal and prime ideals, quotients.
- Homomorphisms and isomorphisms of rings.
- Factorization in domains, Euclidean domains, principal ideal domains, unique factorization domains.
- Recapitulation of vector spaces and linear transformations, eigenvalues and eigenvectors, diagonalization.
- Modules, direct sums, free modules.
- Quotients and momomorphisms of modules, simple modules.
- Modules over principal ideal domains.
- Invariant subspaces for a linear transformation, simultaneous triangulation and diagonalization, Jordan decomposition of a linear transformation.
- Rational and Jordan canonical forms of matrices.
- Inner product spaces, The Gram-Schmidt orthogonalization. orthogonal complements.
- The adjoint of a linear operator, normal and self-adjoint operators, Unitary and orthogonal operators, orthogonal projections and spectral Theorem.

Recommended Reading

1. M Artin, Algebra, Prentice-Hall of India, New Delhi (1994).
2. K R Hoffman and R A Kunze, Linear Algebra, Pearson Education (1971).
3. C Musili, Introduction to Rings and Modules, Narosa Publishing House (1994).
4. N S Gopalakrishnan, University Algebra, New Age International (1986).

5. Nathan Jacobson, Basic Algebra Vol. I, Dover Publications (2009).
6. I S Luthar and I B S Passi, Algebra Vol. II & III, Narosa Publishing House, New Delhi (2002).

Grading Policy

We will have 2 mid-semester examinations worth 20 marks each. There will be a quiz during most tutorial sessions worth 4 marks; this quiz will be based on assignment questions given during that week. The best five quizzes will be counted towards another 20 marks. The final end-semester examination will be worth 40 marks out of which 10 marks will be entirely based on the assignments.

1st Mid Sem	20
2nd Mid Sem	20
Quizzes	20
End Sem	40
Total	100

In each assignment, there will be some challenging (starred) problems. Solving these problems should be its own reward! However, if you do solve these problems you will be better prepared to do the challenging portion of the end-semester examination (which will be worth about 20-30 marks). Elegant and/or original solutions to the challenging problems will get a special mention.

What the course is about

What is a number? One answer to this question is similar to the answer to the question "What is a duck?" which is, "If it walks like a duck and quacks like a duck, then it is a duck." Algebra is the study of number-like things (which we think of as numbers) and systems of such numbers.

We note in passing that mathematics is a lot about taking some familiar objects, abstracting their properties, and then looking for other contexts in which these properties hold.

So we can look for mathematical objects that behave like numbers. In other words, the "usual" operations of addition and multiplication should be defined and behave like they do for numbers. Before we do this, therefore, we need to enumerate the properties that we are looking for. At the very least we have:

- Addition is commutative and associative.

$$a + b = b + a; (a + b) + c = a + (b + c)$$

- Multiplication is commutative and associative.

$$a \cdot b = b \cdot a; (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

- Multiplication distributes over addition.

$$a \cdot (b + c) = a \cdot b + a \cdot c; (a + b) \cdot c = a \cdot c + b \cdot c$$

These may seem “trivial” and “obvious” but as we shall see repeatedly Mathematics is in good part about making the obvious explicit!

In addition to these properties, we have subtraction (for integers) and division (for fractions or rational numbers). Since we already have a background in group theory, let us recall that subtraction is the same as addition of the “additive inverse” of a number. The property of subtraction can be stated as follows:

- Addition has an identity element 0.

$$a + 0 = a$$

- Every number a has an additive inverse $-a$.

$$a + (-a) = 0$$

Similarly, division is the same as multiplication by the “multiplicative inverse”. We will not write it out since division is not a property that we wish to insist on in general. Similarly, we will not insist on multiplication being commutative in general.

Mathematics was originally thought to be about “universal” truths. Hence, there was only *one* geometry which was expected to be “correct” and only *one* number system. It was therefore natural to write “axioms” for this *one* true mathematics.

Today, we are more circumspect and realise that there are many truths dependent upon the context. It is therefore better to define *a* geometrical structure like a (topological or other) space and *an* algebraic structure like a ring. You will see this a lot in all your mathematics courses: mathematical objects are “sets with structure”.

A *ring* is a set R with two operations: addition $(+)$ and multiplication $(\cdot)$.

- Addition and multiplication are associative.
- Multiplication distributes over addition.
- Addition is commutative.
- There is an additive identity 0 and a multiplicative identity 1.
- Every element a of R has an additive inverse denoted by $-a$ and we use $a - b$ in place of $a + (-b)$.

Note that we do not (in general) insist that 1 is different from 0! As an exercise, show that $1 = 0$ if and only if $R = \{0\}$ is the ring that contains only one element su that $0 + 0 = 0$ and $0 \cdot 0 = 0$. This is the zero ring!

Exercise: Show that $0 = 1$ in a ring if and only if the ring consists of just one element 0 with $0 + 0 = 0$ and $0 \cdot 0 = 0$.

Exercise: In a ring R show that for any element a we have $a \cdot 0 = 0$.

The fundamental example of a ring is $\mathbb{Z}$ which consists of all the integers. As we shall see it is the "one ring that binds them all."