(1) $$\frac{\dot{a}^2}{a^2} = H_0^2\left[\Omega_{nr}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda + (1-\Omega_{nr}-\Omega_\Lambda)\frac{a_0^2}{a^2}\right]$$

$$\frac{\ddot{a}}{a} = H_0^2\left[\Omega_\Lambda - \frac{1}{2}\Omega_{nr}\left(\frac{a_0}{a}\right)^3\right]$$

(a) minimum $\frac{a}{a}$, ~~such~~ such that $\frac{a_{min}}{a_0} > 0$

requires $\dot{a} = 0$, $\ddot{a} > 0$ @ $a = a_{min}$.

$$\Rightarrow \Omega_{nr}\left(\frac{a_0}{a_{min}}\right)^3 + \Omega_\Lambda + (1-\Omega_{nr}-\Omega_\Lambda)\left(\frac{a_0}{a_{min}}\right)^2 = 0$$

$\Rightarrow \Omega_{nr} + \Omega_\Lambda > 1$, otherwise this is not possible.

$$\ddot{a} > 0 \Rightarrow \Omega_\Lambda - \frac{1}{2}\Omega_{nr}\left(\frac{a_0}{a}\right)^3 > 0$$

$$\Omega_{nr}\left(\frac{a_0}{a_{min}}\right)^3 = (\Omega_{nr}+\Omega_\Lambda-1)\left(\frac{a_0}{a_{min}}\right)^2 - \Omega_\Lambda$$

$$\Rightarrow \Omega_\Lambda - \frac{1}{2}\left[(\Omega_{nr}+\Omega_\Lambda-1)\left(\frac{a_0}{a_{min}}\right)^2 - \Omega_\Lambda\right] > 0$$

$$\Rightarrow \frac{3\Omega_\Lambda}{2} > \frac{1}{2}(\Omega_{nr}+\Omega_\Lambda-1)\left(\frac{a_0}{a_{min}}\right)^2$$

$$\Rightarrow \Omega_\Lambda > \frac{1}{3}(\Omega_{nr}+\Omega_\Lambda-1)\left(\frac{a_0}{a_{min}}\right)^2$$

(b) $\Omega_{m} + \Omega_{\Lambda} > 1$

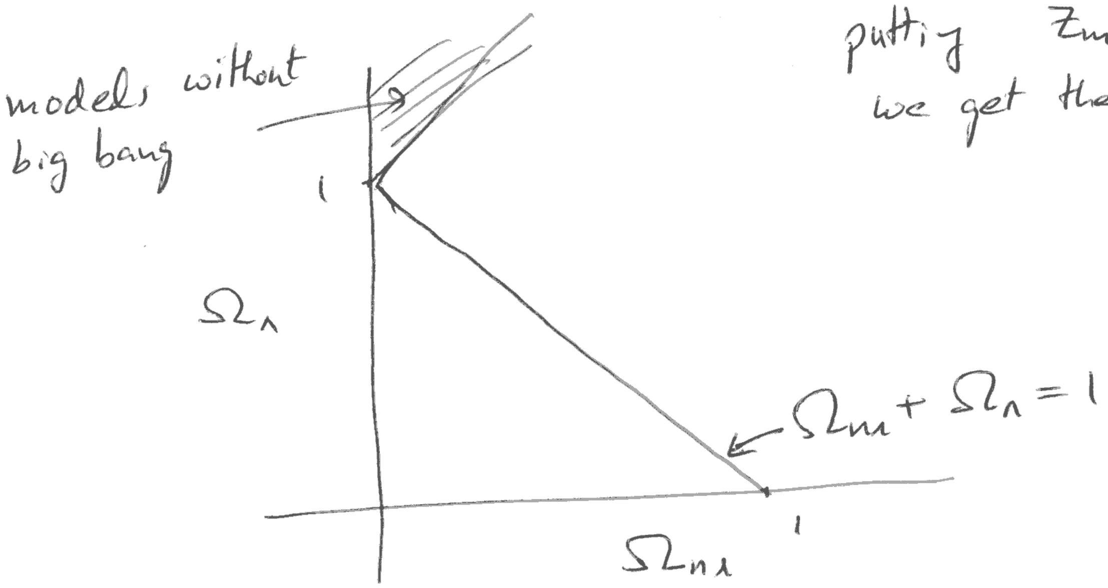
and

$$\Omega_{\Lambda} > \frac{1}{3}\left(\Omega_{m} + \Omega_{\Lambda} - 1\right)\left(\frac{a_0}{a_{min}}\right)^2$$

$$= \frac{1}{3}\left(\Omega_{m} + \Omega_{\Lambda} - 1\right)\left(1 + z_{max}\right)^2$$

here z_{max} is the highest redshift possible.

putting $z_{max} = \epsilon$, we get the envelope



(c) Conditions for this are :

$$\dot{a} = 0, \quad \ddot{a} < 0$$

$\Rightarrow \quad \Omega_{nr} + \Omega_\Lambda - 1 \geq 0$ is required.

$$\Omega_\Lambda - \frac{1}{2}\Omega_{nr}\left(\frac{a_0}{a_{max}}\right)^3 < 0$$

$$\Rightarrow \quad \Omega_\Lambda < \frac{1}{2}\Omega_{nr}\left(\frac{a_0}{a_{max}}\right)^3$$

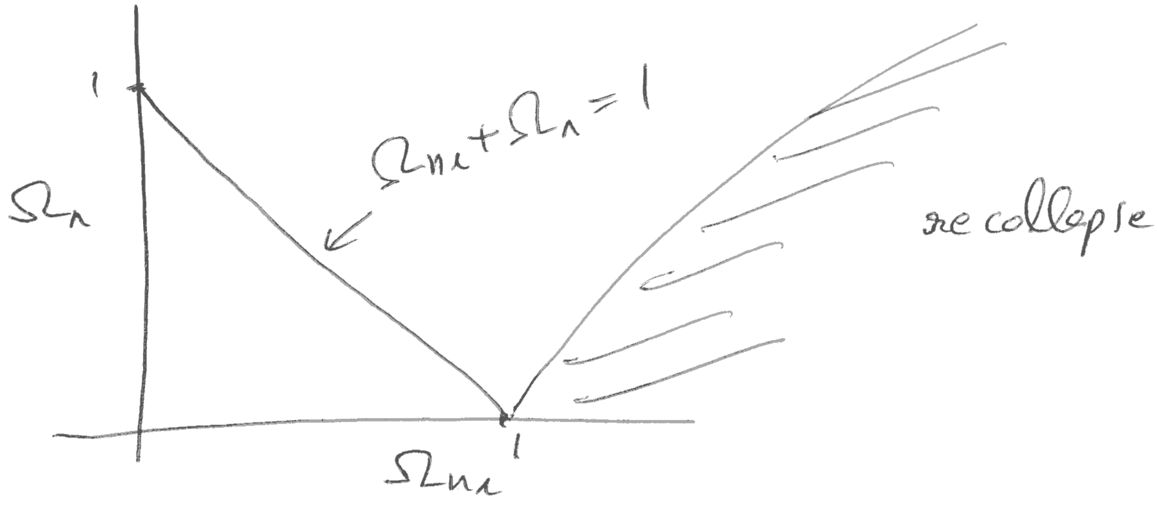
$$\Rightarrow \quad \Omega_{nr} > 2\Omega_\Lambda\left(\frac{a_{max}}{a_0}\right)^3$$

(d) $$\Omega_{nr} + \Omega_\Lambda > 1$$

and

$$\Omega_{nr} > 2\Omega_\Lambda\left(\frac{a_{max}}{a_0}\right)^3$$

putting $a_{max} = a_0 + \epsilon$, we get.



(1e)

$$\Omega_{m0}\left(\frac{a_0}{a_{eq}}\right)^3 = \Omega_\Lambda$$

$$\Rightarrow \quad (1+z_{eq})^3 = \frac{\Omega_\Lambda}{\Omega_{m0}}$$

$$\Rightarrow \quad (1+z_{eq}) = \left(\frac{\Omega_\Lambda}{\Omega_{m0}}\right)^{1/3}$$

$$\Rightarrow \quad z_{eq} = \left(\frac{\Omega_\Lambda}{\Omega_{m0}}\right)^{1/3} - 1$$

$$= \left(\frac{0.8}{0.2}\right)^{1/3} - 1 = 4^{1/3} - 1$$

$$= 1.59 - 1$$

$$\Rightarrow \quad z_{eq} = 0.59.$$

(1f)

$$\ddot{a} \geq 0 \quad \Rightarrow \quad \Omega_\Lambda - \frac{1}{2}\Omega_{m0}\left(\frac{a_0}{a}\right)^3 \geq 0$$

$$\Rightarrow \quad (1+z_{acc})^3 \leq \frac{2\Omega_\Lambda}{\Omega_{m0}}$$

$$\Rightarrow \quad z_{acc} = \left(\frac{2\Omega_\Lambda}{\Omega_{m0}}\right)^{1/3} - 1$$

$$\Rightarrow \quad \boxed{z_{acc} = 1}$$

(2a)

$$n_{gal} = \int_0^\infty \Phi(L)\, dL$$

$$= \Phi_* \int_0^\infty \left(\frac{L}{L_*}\right)^{-\alpha} e^{-L/L_*} \frac{dL}{L_*}$$

$$= \Phi_* \Gamma(1-\alpha)$$

(2b)

$$l_{gal} = \int_0^\infty L\, \Phi(L)\, dL$$

$$= \Phi_* L_* \int_0^\infty \left(\frac{L}{L_*}\right)^{1-\alpha} e^{-L/L_*} \frac{dL}{L_*}$$

$$= \Phi_* L_* \Gamma(2-\alpha)$$

(2c)

$$\rho_{gal} = l_{gal} \frac{M}{L} = l_{gal} . 5 \frac{M_\odot}{L_\odot}$$

$$\frac{M_\odot}{L_\odot} = \frac{2 \times 10^{30} \text{ Kg}}{4 \times 10^{26} \text{ W}} = 5 \times 10^{3} \text{ Kg/W}$$

$$\Rightarrow \rho_{gal} = 2.5 \times 10^{4}\, l_{gal}$$

(2d) $$\rho_c = \frac{3H_0^2}{8\pi G}$$

$$= \frac{3 \times (70 \times 10^3 / 3.08 \times 10^{22})^2}{8 \times 3.14 \times 6.7 \times 10^{-11}}$$

$$= \frac{3 \times \left(\frac{7}{3.08}\right)^2 \times 10^{-36}}{8 \times 3.14 \times 6.7 \times 10^{-11}}$$

$$= 9.2 \times 10^{-27} \ \mathrm{kg/m^3}.$$

(2e) $$\Phi_* = 5 \times 10^{-3} \ \mathrm{Mpc}^{-3}$$

$$\alpha = 1.1$$

$$L_* = 2.4 \times 10^{10} L_\odot = 9.6 \times 10^{36} \ \mathrm{W}$$

$$\Rightarrow n_{gal} = \Phi_* \Gamma(1-\alpha)$$

$$= 5 \times 10^{-3} \times \Gamma(-0.1)$$

$$= 5.35 \times 10^{-2} \ \mathrm{Mpc}^{-3}.$$

$$l_{gal} = \Phi_* L_* \Gamma(2-\alpha)$$

$$= 5 \times 10^{-3} \times 2.4 \times 10^{10} L_\odot \Gamma(0.9)$$

$$= 1.2 \times 10^8 \times 1.07 \times L_\odot / \mathrm{Mpc}^3$$

$$= 1.3 \times 10^8 \ L_\odot / \mathrm{Mpc}^3$$

$$\rho_{gal} = 2.5 \times 10^{4}\, \ell_{gal}$$

$$= 2.5 \times 10^{4} \times 1.3 \times 10^{8} \times 4 \times 10^{26} \,/\mathrm{Mpc}^{3}$$

$$= 1.3 \times 10^{38}\ \mathrm{kg}/\mathrm{Mpc}^{3}$$

$$= 6.5 \times 10^{7}\ M_{\odot}/\mathrm{Mpc}^{3}$$

$$= 4.4 \times 10^{-30}\ \mathrm{kg}/\mathrm{m}^{3}$$

$$\Omega_{gal} = \frac{\rho_{gal}}{\rho_c} = \frac{4.4 \times 10^{-30}}{9.2 \times 10^{-27}} = 4.8 \times 10^{-4}$$

(3)

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G \bar{\rho}_{m} \delta = 0$$

(a) $\Omega_m = 1, \quad \Omega_\Lambda = 0.$

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left(\frac{a_0}{a}\right)^3$$

$$y = \frac{a}{a_0}, \quad x = t H_0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{y}$$

$$\Rightarrow y^{1/2} dy = dx$$

$$\Rightarrow \frac{y^{3/2}}{3/2} = x + \text{const.}$$

if we choose, $a = 0$ @ $t = 0$ then

$y = 0$ @ $x = 0$

$\Rightarrow$ const. $= 0$

$$\Rightarrow y = \left(\frac{3x}{2}\right)^{2/3}$$

$$a = a_0 \left(\frac{3}{2} t H_0\right)^{2/3}$$

but @ $t = t_0$, $a = a_0$

$$\Rightarrow \frac{3}{2} t_0 H_0 = 1 \Rightarrow \boxed{a = a_0 \left(\frac{t}{t_0}\right)^{2/3}}$$

(3b)

$$a = a_0 \left(\frac{t}{t_0}\right)^{2/3}$$

$$\Rightarrow \dot{a} = \frac{2}{3} a_0 t^{-1/3} t_0^{-2/3}$$

$$\Rightarrow \frac{\dot{a}}{a} = \frac{2}{3t}$$

$$\bar{\rho}_{nr} = \bar{\rho}_{nr0} \left(\frac{a_0}{a}\right)^3$$

$$= \rho_c \left(\frac{a_0}{a}\right)^3 = \frac{3H_0^2}{8\pi G} \left(\frac{a_0}{a}\right)^3$$

$$= \frac{3H_0^2}{8\pi G} \left(\frac{t_0}{t}\right)^2$$

$$\Rightarrow 4\pi G \bar{\rho}_{nr} = \frac{3H_0^2}{\cancel{8\pi G}_{\,2}} \frac{t_0^2}{t^2} \cdot \cancel{4\pi G}$$

$$= \frac{3}{2} \cdot \left(\frac{2}{3}\right)^2 \frac{1}{t^2} = \frac{2}{3t^2}$$

$$\Rightarrow \ddot{\delta} + \frac{4}{3t} \dot{\delta} - \frac{2}{3t^2} \delta = 0$$

this has power law solutions.

$\delta \propto t^m$

$$m(m-1) t^{m-2} + \frac{4}{3} m t^{m-2} - \frac{2}{3} t^{m-2} = 0$$

this is true at all t

$$\Rightarrow \quad m(m-1) + \frac{4}{3}m - \frac{2}{3} = 0$$

$$\Rightarrow \quad m^2 + \frac{m}{3} - \frac{2}{3} = 0$$

~~soln~~

$$m(m+1) = m^2 + m$$

$$\Rightarrow \quad m(m+1) - \frac{2}{3}(m+1) = 0$$

$$\Rightarrow \quad \left(m - \frac{2}{3}\right)(m+1) = 0$$

$$\Rightarrow \quad \boxed{\delta = A t^{2/3} + B t^{-1}}$$

(3c)

$$H = \frac{\dot{a}}{a}$$

$$4\pi G \bar{\rho}_{m} = \frac{3}{2} H_0^2 \Omega_{m} \left(\frac{a_0}{a}\right)^3$$

$$\dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2$$

$$= H_0^2 \left[\Omega_\Lambda - \frac{1}{2}\Omega_{m} \left(\frac{a_0}{a}\right)^3\right] - H^2$$

$$\rightarrow \quad \ddot{H} = + \frac{3}{2} \Omega_{m} H_0^2 \left(\frac{a_0}{a}\right)^3 \frac{\dot{a}}{a} - 2 H \dot{H}$$

$$\Rightarrow \quad -\cancel{2H\dot{H}} + \frac{3}{2}\cancel{\Omega_m\left(\frac{a_0}{a}\right)^3 H H_0^2}$$

$$+ \cancel{2H\dot{H}} - \frac{3}{2}\cancel{\Omega_m\left(\frac{a_0}{a}\right)^3 H_0^2 H}$$

$$= 0$$

hence $H = \frac{\dot{a}}{a}$ is a soln.

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(4)

$$\rho(R) = \frac{\rho_{vir} R_{vir}^3}{(R^2 + R_{vir}^2) R}$$

(a) Average mass in $R_{vir} = \frac{4\pi}{3} \bar{\rho}_{nr} R_{vir}^3$

$$= \frac{4\pi}{3} \rho_c (1 + z_{coll})^3 R_{vir}^3$$

we have assumed ~~eathe~~ Einstein-deSitter model.

$$M_{av} = \frac{4\pi}{3} \cdot \frac{3H_0^2}{8\pi G} (1 + z_{coll})^3 R_{vir}^3$$

$$= \frac{4\pi}{3} \rho_c (1 + z_{coll})^3 R_{vir}^3$$

$$M(r) = \int_0^r 4\pi R^2 dR \, \rho(R)$$

$$= 4\pi \rho_{vir} R_{vir}^3 \int_0^r \frac{R \, dR}{R_{vir}^2 + R^2}$$

$$x = \frac{R}{R_{vir}}$$

$$\Rightarrow M(r) = 4\pi \rho_{vir} R_{vir}^3 \int_0^{r/R_{vir}} \frac{x \, dx}{1 + x^2}$$

$$y = 1 + x^2, \quad dy = 2x \, dx.$$

$$\Rightarrow M(r) = 2\pi \rho_{vir} R_{vir}^3 \int_1^{1+r^2/R_{vir}^2} \frac{dy}{y}$$

$$= 2\pi \rho_{vir} R_{vir}^3 \ln\left(1 + \frac{r^2}{R_{vir}^2}\right)$$

$$\Rightarrow M_{vir} = 2\pi \rho_{vir} R_{vir}^3 \ln(2).$$

$$\frac{M_{vir}}{M_{av}} = \cancel{18}\,6\,\pi^2 = \frac{\cancel{2}\pi \rho_{vir} \cancel{R_{vir}^3} \ln(2)}{2\,\frac{\cancel{4}\pi}{\cancel{3}} \rho_c (1+z_{coll})^3 \cancel{R_{vir}^3}}$$

$$\Rightarrow \boxed{\rho_{vir} = \frac{12\pi^2}{\ln(2)} \rho_c (1+z_{coll})^3}$$

(4b)

$$M_{vir} = 2\pi \rho_{vir} \ln(2) R_{vir}^3$$

$$= 2\pi \cancel{\ln(2)} R_{vir}^3 \cdot \frac{12\pi^2 \rho_c (1+z_{coll})^3}{\cancel{\ln(2)}}$$

$$= 24\pi^3 (1+z_{coll})^3 \rho_c R_{vir}^3$$

$$\Rightarrow R_{vir} = \left(\frac{M_{vir}}{24\pi^3 \rho_c}\right)^{1/3} (1+z_{coll})^{-1}$$

(4c)

$$\frac{v_{circ}^2}{R} = \frac{G M(R)}{R^2}$$

$$\Rightarrow v_{circ}^2 = \frac{G M(R)}{R}$$

$$= \frac{24\pi^3 \rho_c (1+z_{coll})^3 \cdot G\left[\ln\left(1+\frac{R^2}{R_{vir}^2}\right)\right] R_{vir}^3}{R \ln(2)}$$

for $R \ll R_{vir}$ $\quad \ln\left(1+\frac{R^2}{R_{vir}^2}\right) \simeq \frac{R^2}{R_{vir}^2}$

$$v_{circ}^2 = \frac{24\pi^3 \rho_c (1+z_{coll})^3 G R_{vir}^2 \cdot R}{\ln(2)}$$

$$= \frac{G M_{vir}}{\ln(2)} \cdot \frac{R}{R_{vir}^2}$$

for $R \simeq R_{vir}$

$$v_{circ}^2 = \frac{G M_{vir}}{R_{vir}} \cancel{\frac{G M_{vir}}{\ln(2) R_{vir}}\left(1 - \frac{R}{R_{vir}}\right)}$$

(4d) $$\Phi = -\frac{G M(R)}{R}$$

$$= -\frac{G M_{vir}}{\ln(2)\, R} \ln\left(1 + \frac{R^2}{R_{vir}^2}\right)$$

(4e) $$\frac{3}{2} k_B T_{vir} N = \frac{3}{5} \frac{G M_{vir}^2}{R_{vir}}$$

$$\Rightarrow T_{vir} = \frac{2}{5} \cdot \frac{G M_{vir} \mu m_p}{k_B R_{vir}}$$

$$= \frac{2}{5} \cdot \frac{\mu m_p G}{k_B} \frac{M_{vir}}{R_{vir}}$$

$$= \frac{2}{5} \frac{\mu m_p G}{k_B} \cdot M_{vir}^{2/3} \cdot \left(24 \pi^3 \rho_c (1 + z_{coll})^3\right)^{1/3}$$

(4f) $$\rho_{gas}(R) = \rho_{gas}(0)\, e^{-\Phi(R) \mu m_p / k_B T}$$