Indian Institute of Science Education and Research, Mohali
Cosmology and Galaxy Formation (PHY654)
(January – April 2016)
Problem Set 3

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1. Quasar spectra:
   (a) Using the Rydberg formula, calculate the wavelength of the first three Lyman series transitions for Hydrogen.
   (b) Given a quasar at $z = 3.5$, calculate the observed wavelength of its Lyman-α emission.
   (c) The spectrum of the quasar shows a number of absorption lines due to neutral Hydrogen in the intervening space between the quasar and us. As any matter in the intervening space is expected to have $z < z_{quasar}$, find out the range of redshifts for which the Lyman-α absorption is not mixed with Lyman-β absorption from other redshifts?
   (d) Carbon ion C^{+++}, or CIV as it is written in the spectroscopic notation, has a line at 1549 Å. This line is also seen in absorption in Quasar spectra. Find out the range of redshifts where this line will begin to mix with Lyman-α absorption from other redshifts at $z < z_{quasar}$.
2. Consider a distribution of N particles, each of mass m. Locations of these particles are r_i and the particles are distributed in a compact region.
   (a) Write down an expression for the gravitational potential due to this mass distribution at a distant point.
   (b) Carry out a multipole expansion for potential.
   (c) Show that if we choose the centre of mass of the distribution as the origin then the dipole moment vanishes.
   (d) Write down the corresponding multipole expansion for the acceleration.
   (e) Write down an expression for the first two non-vanishing moments in terms of particle positions.
   (f) If we only retain the first term then the next term is the dominant contribution to the error. If the size of the mass distribution is D and the distance from the point where acceleration is to be calculated is $R \gg D$ then estimate the magnitude of the error.
   (g) Estimate the number of operations required to calculate the first two moments for N particles. Compare with the number of operations required for calculating the force due to each particle individually.
3. Consider the Poisson equation in Fourier space:

$$\varphi_k = -\frac{4\pi G \varrho_k}{k^2}$$

If we divide the potential into two parts along the following lines:

$$\begin{aligned}
\varphi_k &= -\frac{4\pi G\varrho_k}{k^2} \\
&= -\frac{4\pi G\varrho_k}{k^2}\exp\left(-k^2 r_s^2\right) - \frac{4\pi G\varrho_k}{k^2}\left(1-\exp\left(-k^2 r_s^2\right)\right) \\
&= \varphi_k^l + \varphi_k^s \\
\varphi_k^l &= -\frac{4\pi G\varrho_k}{k^2}\exp\left(-k^2 r_s^2\right) \\
\varphi_k^s &= -\frac{4\pi G\varrho_k}{k^2}\left(1-\exp\left(-k^2 r_s^2\right)\right)
\end{aligned}$$

Here r_s is a transition scale.

(a) Show that the short range force per unit mass due to a particle of mass m is:

$$\mathbf{f}^s(\mathbf{r}) = -\frac{Gm\mathbf{r}}{r^3}\left(\mathrm{erfc}\left(\frac{r}{2r_s}\right) + \frac{r}{r_s\sqrt{\pi}}\exp\left(-\frac{r^2}{4r_s^2}\right)\right)$$

Here erfc is the complementary error function.

(b) Write down an expression for the corresponding long range force.

(c) For $r_s = 1$, plot the short range, the long range and the total force as a function of r.

(d) For $r_s = 1$, find out the scale where the short range force is 1% of the total force? At what scale does it become 0.1%?

(e) Repeat the last exercise for $r_s = 2$.

4. Neighbours.

(a) Show that in a random distribution of particles with number density $\bar{n}$, the average number of neighbours a particle has between r and $r + dr$ is $4\pi r^2 \bar{n} dr$.

(b) The two point correlation function $\xi(r)$ is defined in terms of the average number of neighbours for a particle in the given distribution:

$$N(r)dr = 4\pi r^2 \bar{n} dr \left(1 + \xi(r)\right)$$

Here $N(r)dr$ is the average number of neighbours between r and $r + dr$ and $\bar{n}$ is the average number density of particles. Find out the number of neighbours up to a distance r.

(c) If $\bar{\xi}(r)$ is the fractional excess in number of neighbours up to a distance r then write an expression for $\bar{\xi}(r)$ in terms of the two point correlation function $\xi(r)$.

5. Junction conditions for a shock are defined as:

$$\begin{aligned}
j \equiv \varrho_1 v_{1x} &= \varrho_2 v_{2x} \\
p_1 + \varrho_1 v_{1x}^2 &= p_2 + \varrho_2 v_{2x}^2 \\
\varrho_1 v_{1x} v_{1y} &= \varrho_2 v_{2x} v_{2y} \\
\varrho_1 v_{1x} v_{1z} &= \varrho_2 v_{2x} v_{2z} \\
\varrho_1 v_{1x}\left[h_1 + \frac{1}{2}v_1^2\right] &= \varrho_2 v_{2x}\left[h_2 + \frac{1}{2}v_2^2\right]
\end{aligned}$$

Here ϱ_1 is the density of the fluid before the shock, and ϱ_2 is the post-shock density. v_1 and v_2 specify the speed of the fluid, before and after the shock respectively. A further subscript on this denotes a specific component. We have assumed that the shock is in the $y - z$ plane. p_1 and p_2 denote pressure, before and after the shock, respectively. The Enthalpy is denoted by h_1 and h_2.

(a) Show that we can have a shock where the mass flux j through the discontinuity is zero. This is called a tangential discontinuity. Show that in this case v_y, v_z and ϱ can be discontinuous but not p or v_x.

(b) For the case where $j \neq 0$, show that v_y and v_z are continuous. Use this to simplify the junction condition for energy flux.

(c) Using the definition of the specific volume: $V_1 = 1/\varrho_1$, $V_2 = 1/\varrho_2$, write down the mass flux in terms of the pressure and specific volume on two sides of the junction.

(d) Use this to obtain an expression for the change in the normal component of velocity, i.e., $v_{1x} - v_{2x}$ in terms of the pressure and specific volume on two sides of the shock.

(e) Use the above relations to obtain the discontinuity in Enthalpy across the shock in terms of the pressure and specific volume.

(f) Write down an expression for discontinuity in the internal energy.

(g) If you are given that the equation of state for the gas is $pV^\gamma = constant$, $h = \gamma pV/\left(\gamma - 1\right)$. Obtain a relation for V_2 in terms of V_1, p_1 and p_2. What is the limiting form if $p_2 \gg p_1$?

(h) Write down an expression for the speed of sound in terms of Enthalpy and γ. Use this to convert velocities to Mach number on either side of the shock.

(i) For ideal gas, obtain an expression for the ratio of temperature on the two sides of the interface in terms of the Mach number of fluid on two sides of the shock.