Indian Institute of Science Education and Research, Mohali

Cosmology and Galaxy Formation (PHY654)

(January – April 2016)

Problem Set 2

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1. Write down the continuity, the Poisson and the Navier-Stokes equation for a fluid in proper/physical coordinates.
$$\left(\frac{\partial \varrho}{\partial t}\right)_{\mathbf{R}} + \left(\frac{\partial}{\partial \mathbf{R}}\right)_t \left[\varrho \frac{d\mathbf{R}}{dt}\right] = 0$$
Here $\varrho(\mathbf{R}, t)$ is the density, t is the usual cosmological time and $\mathbf{R}$ is the proper/physical coordinate. If we write $\mathbf{V} = d\mathbf{R}/dt$ then the Navier-Stokes equation takes the form:
$$\left(\frac{\partial \mathbf{V}}{\partial t}\right)_{\mathbf{R}} + \left(\mathbf{V}.\frac{\partial}{\partial \mathbf{R}}\right)_t \mathbf{V} = -\left(\frac{\partial}{\partial \mathbf{R}}\right)_t \Phi$$
Here Φ is the gravitational potential. The Poisson equation is:
$$\left(\frac{\partial^2}{\partial \mathbf{R}^2}\right)_t \Phi = 4\pi G \varrho$$
Transform to comoving coordinates and write equations for the density contrast, peculiar velocity and the gravitational potential due to density perturbations. Here the comoving coordinates $\mathbf{r}$ are related to physical coordinate as $\mathbf{R} = a(t)\mathbf{r}$ with $a(t)$ the scale factor. Peculiar velocity $\mathbf{u}$ here is defined as: $\mathbf{V} = (\dot{a}/a)\mathbf{R} + \mathbf{u}$. Potential due to perturbations is defined as $\varphi = \Phi - \bar{\Phi}$. Here $\bar{\Phi} = -(1/2)(\ddot{a}/a)R^2$. Density contrast is defined as $\varrho(\mathbf{R}, t) \equiv \bar{\varrho}(t)\left(1 + \delta(\mathbf{R}, t)\right)$ with $\bar{\varrho}$ being the average density of the universe.

2. **Difficult Question.** Can you combine these equations in the special case of a spherically symmetric system with only radial motions and write down a non-linear differential equation for the density contrast? Do you have to make any other assumptions in the calculation?

3. Write down the linearized form of the equations in comoving coordinates.

4. Linear perturbation theory.
   (a) Combine the three equations to derive a second order ordinary differential for the density contrast δ.
   (b) Solve the equation for δ in the Einstein-deSitter universe and show that the two solutions are proportional to $a(t)$ and $H(t)$.
   (c) Show that for a universe with non-relativistic matter, curvature and cosmological constant, $H(t)$ is the decaying solution for δ.
   (d) Derive the expression for the growing mode using the method of Wronskian.

(e) Numerically calculate the solution for the growing mode $D_+(t)$ for a universe with only matter and curvature and show that:

$$f \equiv \frac{d \ln D_+}{d \ln a} \simeq \Omega_{nr}^{0.6}$$

Can you derive a better fit?

(f) Numerically calculate the solution for the growing mode $D_+(t)$ for a universe with only matter and cosmological constant. Compute an approximate form for f in terms of Ω_{nr} and Ω_Λ.

(g) Find out the component of initial conditions that contributes to the decaying mode. This is a function of the initial potential and velocities. Also find the corresponding expression for the growing mode.

(h) What fraction of the initial density contrast contributes to the growing mode if initial peculiar velocities are zero?

(i) How must we set the initial velocities to ensure that the system is entirely in the growing mode?

5. Zel'dovich approximation:

(a) Using linearized equations, show that $d\mathbf{r}/dD_+$ is constant for a system in the growing mode. Use the Lagrangian interpretation of this to write down an equation describing trajectories for fluid elements. This is known as the Zel'dovich approximation if used beyond the linear regime.

(b) Assuming a one to one mapping from initial conditions to final conditions, write down an expression for density contrast by using the trajectories defined above.

6. Spherical Collapse:

(a) Consider a spherically symmetric perturbation in proper coordinates. Write down the equation of motion and derive the first integral with the assumption that the mass enclosed is constant. Assuming that the initial peculiar velocity is zero, write down an expression for the first integral.

(b) For a universe with only non-relativistic matter and curvature, write down the condition for the perturbation to be bound.

(c) Solve the equations for the case a bound perturbation in an Einstein-deSitter universe.

(d) Use constancy of mass within a given shell to write down an expression for the density contrast.

(e) Write down an expression for the scale factor in terms of the parametric solution for time. Also write the limiting form at early times.

(f) Write down an expression for the density contrast at early times. Write it in terms of the scale factor and interpret the coefficients.

(g) Write down the energy at the time when the perturbation reaches its maximum radius. Write an expression for the maximum radius in terms of the initial radius and density contrast. Calculate the time at which the perturbation reaches the maximum radius t_{ta}. Calculate the non-linear density contrast and the linearly evolved density contrast at the time when the perturbation is at the maximum radius.

(h) Apply the condition for virial equilibrium and derive an expression for the radius of the perturbation once it reaches an equilibrium state.

(i) Assuming that the perturbation reaches virial equilibrium at time $t = 2t_{ta}$, calculate the linearly evolved and the non-linear density contrast at this time.

(j) If we assume that the radius of the shell does not evolve after reaching virial equilibrium, how will the density contrast of the perturbation evolve after this time.

7. Isothermal halos. An isothermal halo has a density profile $\rho(R) = \rho_{vir} R_{vir}^2 / R^2$. Assume that the background universe is described by the Einstein-deSitter model.

   (a) Calculate the radius of an isothermal halo of mass M that collapses at redshift z_{coll}.

   (b) Calculate the circular velocity V_{circ} for such a halo.

   (c) Calculate the virial temperature T_{vir} for such a halo.

   (d) Plot numerical values of these quantities for $M = 10^{10}$, 10^{12} and 10^{15} $M_\odot$ and $z_{coll} < 10$.

   (e) The Galaxy is expected to have formed at $z_{coll} \simeq 3$ and has a circular velocity 220 km/s. Estimate its total halo mass.

8. Consider the universal density profile proposed by Navarro-Frenk-White (NFW).

$$\varrho(R) = \frac{\varrho_{vir} R_{vir}^3}{\left(R_{vir}^2 + R^2\right) R}$$

   (a) Solve for the circular velocity as a function of R in this halo. Plot the circular velocity as a function of R between $0.01 R_{vir} \leq R \leq R_{vir}$.

   (b) Solve for the gravitational potential in the halo.

   (c) If the halo contains gas in equilibrium at temperature T then write down an expression for the density of gas as a function of radius R. Plot the total density and the gas density as a function of R between $0.01 R_{vir} \leq R \leq R_{vir}$.

9. Random walks.

   (a) Consider a one dimensional random walk with equal steps of size l. A step in either the forward or the backward direction is equally likely. Calculate the distribution of displacements after $N \gg 1$ steps. Use the Stirling approximation to derive the *rms* displacement after N steps.

   (b) Calculate the conditional probability for the displacement to be nl after N steps if the displacement was ml after $M < N$ steps.

   (c) Show that the random walks can be described as a diffusion process with a Fokker-Planck equation with the number of steps and the net displacement as two independent variables.

   (d) Write down an expression for the distribution of displacements if we have an absorbing barrier at ql.

   (e) Calculate the absorption rate at the absorbing barrier as a function of N.

10. Mass functions from random walks.

(a) We can work with Fourier transform of the initial or linearly extrapolated density contrast. For a Gaussian random field, each infinitesimal range of wave numbers contributes a random displacement with zero mean and a given *rms* σ that is a function of the wave number. An object is said to collapse and form a halo if a random walk reaches the absorbing barrier ($\delta_c = 1.686$) for the first time at this mass scale. Use the results from the last problem to write down an expression for number density of collapsed halos in the mass range M to $M + dM$.

(b) Write down an expression for the density of matter inside halos of mass equal to or greater than M.

(c) Write down the conditional probability that a trajectory that reaches the absorbing barrier at mass M was at some density contrast δ_0 at $M' \gg M$.

11. Spectral lines.

(a) Thermal broadening of spectral lines is described as a Gaussian. Write down an expression for thermal broadening of a line centered at frequency ν_0. Calculate the numerical value of thermal width (*rms width*) of the Lyman-α transition for Hydrogen for $T = 10^4$ K.

(b) Natural width of a spectral line is described by a Lorentzian. Calculate the *rms* of a Lorentzian.

(c) The actual line profile is a convolution of the natural and thermal broadening. Write down an expression for the line profile, also known as the Voigt profile. Do the integral numerically and plot the line profile for different combinations of natural and thermal broadening.

12. In presence of absorption, the intensity changes after passage through an absorber as $I_\nu = I_0 \exp\left[-\tau_\nu\right]$ where $\tau\nu$ is the optical depth at frequency ν. For a spectral line absorption, the frequency variation of τ_ν around the line centre is given by the Voigt profile. The equivalent width of absorption W is defined as:

$$W = \int\limits_{-\infty}^{\infty} d\nu \left(1 - \exp\left[-\tau_\nu\right]\right)$$

The absorption at the line centre is proportional to the number of absorbers.

(a) Calculate the equivalent width for a spectral line that is optically thin, i.e., $\tau_\nu \ll 1$. Do this for a line with a Gaussian profile as well as a Lorentzian profile. How does the equivalent width change with the number of absorbers N (also called column density)?

(b) Calculate the variation of equivalent width W with N for the Voigt profile and plot the result. By plotting these on a log-log plot, obtain the characteristic variation of W with N in different regimes.