Solutions to Assignment 10

1. We throw a fair die 100 times looking for a '6'.
   (a) Write a formula for the probability that there are there are at least 50 and at most 60 '6's.

   Solution: The random variable X that counts the number of heads follows the Binomial distribution. Hence,

   $$P(50 \leq X \leq 60) = \frac{1}{6^{100}} \sum_{r=50}^{60} \binom{100}{r} 5^{100-r}$$

   (b) Write an approximate formula for the above probability as an integral.

   Solution: We use the de Moivre-Laplace approximation or Central Limit Theorem to get the approximation

   $$P(a \leq (X-m)/\sigma \leq b) \simeq \frac{1}{\sqrt{2\pi}} \int_a^b \exp(-s^2/2) ds$$

   Here $m = 100/6$ and $\sigma^2 = 500/36$ so we get

   $$P(50 \leq X \leq 60) \simeq \frac{1}{\sqrt{2\pi}} \int_a^b \exp(-s^2/2) ds$$

   where $a = (50 - 50/3)/(5\sqrt{5}/3) = 4\sqrt{5}$ and $b = (60 - 50/3)/(5\sqrt{5}/3) = 26/\sqrt{5}$.

2. Estimate the following numbers. In each case, first define the number by a series or sequence and then *prove* the relevant convergence.

   $$\pi; \log(2); \exp(1); \sqrt{10}; \exp(-0.5)$$

   Solution: We have the series $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$ is an alternating series. Hence the error is the same as the first term which is dropped. To get the value with error less than 1/11 we can add the terms

   $$\pi/4 \simeq 1 - 1/3 + 1/5 - 1/7 + 1/9 = 263/319$$

   This says that π lies in the range $[1052/319 - 4/11, 1052/319 + 4/11]$.

Similarly, we have the series $\log(2) = 1 - 1/2 + 1/3 + \dots$. However, in this case we know that

$$\log(2) = -\log(1/2) = -log(1 - 1/2) = 1/2 + (1/2)^2/2 + (1/2)^3/3 + \dots$$

This series converges much faster since the error on dropping all terms after k is at most $1/2^k k = 1/(2^{k+1}k) + 1/(2^{k+2}k) + \dots$. So we get that upto an error of at most $1/24$

$$\log(2) \simeq 1/2 + 1/8 + 1/24 = 2/3$$

Hence, $\log(2)$ lies in the interval $[2/3 - 1/24, 2/3 + 1/24]$.

We have the series $\exp(1) = 1 + 1 + 1/2 + 1/6 + 1/24 + \dots$. The sum of all terms after the k-th term is at most $1/k! = (1/k!)(1/2 + 1/4 + \dots)$ as long as $k \geq 1$. Hence, upto an error of at most $1/120$ we have

$$\exp(1) \simeq 1 + 1 + 1/2 + 1/6 + 1/24 + 1/120 = 163/60$$

Hence $\exp(1)$ lies in the interval $[163/60 - 1/120, 163/60 + 1/120]$.

We have $3^2 < 10$, so $a_0 = 3 < \sqrt{10}$ and $\sqrt{10} < 10/a_0 = 10/3 = b_0$. We can improve on this by taking c to be the average of a_0 and b_0 which is $19/6$. Now $c^2 = 361/36 > 10$. So we take $b_1 = 19/6$ and $a_1 = 10/b_1 = 60/19$. We then have $\sqrt{10}$ lying in the interval $[60/19, 19/6]$. Note that this interval has length $1/(6 * 19)$ which is quite small!

We have the alternating series $\exp(-1/2) = 1 - 1/2 + 1/8 - 1/48 + \dots$. The error by dropping the k-term onwards is at most the size of the k-th term. So, up to an error of $1/384$ we have

$$\exp(-1/2) = 1 - 1/2 + 1/8 - 1/48 = 29/48$$

Hence $\exp(-1/2)$ lies in the interval $29/48 - 1/384, 29/48 + 1/384]$.

3. We repeatedly flip two fair coins. Let X_i denote the random variable that takes the value 3 if the i-th double flip returns two Heads and -1 if it returns anything else. Which of the following statements are True? Justify your answer.

   (a) The random variable X_n converges to 0 in probability.

   Solution: False. Since $P(|X_n| \geq 1) = 1$, we see that $P(|X_n| > 1/2)$ does not converge to 0 as n goes to infinity.

   (b) The random variable $W_n = X_n/n$ converges to 0 in probability.

Solution: True. Since $P(|X_n| < 1/2) = 0$, we see that $P(|W_n| < 1/2k) = 0$ for $n > k$.

(c) The random variable $Y_n = (\sum_{i=1}^{n} X_i)/n$ converges to 0 in probability.

Solution: True. Since $E(X_n) = 0$ and $\sigma^2(X) = 1/2$, we have Y_n converges to 0 in probability by the Law of Large Numbers.

(d) The random variable $Z_n = (\sum_{i=1}^{n} X_i)$ converges to 0 in probability.

Solution: False. If Z_n converges to 0 then so does Z_{n-1} and hence so does $X_n = Z_n - Z_{n-1}$. We have already seen in the first part that this is not so.

4. (a) For what values of a and b can the following be the characateristic function of a random variable?

$$a^2 \cos(t) + b^2 \sin(t)/t$$

Solution: We have the condition that $\phi(0) = 1$ if ϕ is to be the characteristic function of a random variable. Thus the conditions is $a^2 + b^2 = 1$.

(b) The characteristic function of a random variable X is given by

$$a \sin(t)/t + b \cos(-3t) + c \exp(2t\sqrt{-1})$$

What are the values of a, b and c for which this has mean 0 and standard deviation 1?

Solution: To be a characteristic function we must have $\phi(0) = 1$. This gives the equation $a + b + c = 1$.

To have the mean 0, we must have $\phi'(0) = 0$, This gives the condition $2c = 0$.

Finally to have standard deviation as 1, we must have second moment as 1 also (since mean is 0). So we must have $\phi''(0) = -1$. This gives the equation $a/3 + 9b + 4c = 1$.

We see that there is no solution for these equations. So this is not possible.