Probability and Statistics
MTH202 Second Mid-Sem Exam 17 March 2016

Write your name and/or registration number in the box provided.
Write your answers in space provided.
You have 1 hour to complete this exam.

Name:____________ *Reg. No:*____________

Question:	1	2	3	4	Total
Points:	3	4	5	3	15
Score:					

1. We flip an unbiased coin 100 times.

(1 mark) (a) Write a formula for the probability that there are there are at least 40 and at most 60 heads.

Solution: The random variable X that counts the number of heads follows the Binomial distribution. Hence,

$$P(40 \leq X \leq 60) = \frac{1}{2^{100}} \sum_{r=40}^{60} \binom{100}{r}$$

(2 marks) (b) Write an approximate formula for the above probability as an integral.

Solution: We use the de Moivre-Laplace approximation or Central Limit Theorem to get the approximation

$$P(a \leq (X-m)/\sigma \leq b) \simeq \frac{1}{\sqrt{2\pi}} \int_a^b \exp(-s^2/2) ds$$

Here $m = 50$ and $\sigma = 5$ so we get

$$P(40 \leq X \leq 60) \simeq \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \exp(-s^2/2) ds$$

2. On the average 5 wickets fall in each innings (20 overs) in a typical T20 cricket match. You watch 10 overs (*corrected to 1 over*) of some match.

(1 mark) (a) Write a formula for the probability that you will see exactly 2 wickets fall.

Solution: The expected number of wickets in 1 over is $c = 5/20 = 1/4$. We will use the Poisson distribution to get

$$P(X = 2) = \frac{c^2}{2!}e^{-c} = \frac{1}{32}e^{-1/4}$$

(1 mark) (b) Write a formula for the probability that you will see at least 1 wicket fall.

Solution: The expected number of wickets in 1 over is $c = 1/4$. We will use the Poisson distribution to get

$$P(X \geq 1) = \sum_{k=1}^{\infty} \frac{c^k}{k!}e^{-c} = 1 - e^{-1/4}$$

(2 marks) (c) Estimate upto 2 places of decimal the probability that you will see no wicket fall.

Solution: As seen above we need to calculate $e^{-1/4} = 1 - P(X \geq 1)$.

$$1 - 1/4 + 1/32 - 1/384 + \dots$$

We see that the terms beyond are less than 1/1000 so we can neglect that and all successive terms since this is an alternating series. So we get

$$(384 - 96 + 12 - 1)/384 = 299/384$$

which is roughly 0.78. So the answer is roughly 0.78 upto two places of decimal.

3. We repeatedly flip a fair coin. Let X_i denote the random variable that takes the value 1 if the i-th flip returns Head and -1 if it returns Tail. Which of the following statements are True? Justify your answer.

(1 mark) (a) The random variable X_n converges to 0 in probability.

Solution: False. Since $P(|X_n| = 1) = 1$, we see that $P(|X_n| > 1/2)$ does not converge to 0 as n goes to infinity.

(1 mark) (b) The random variable $W_n = X_n/n$ converges to 0 in probability.

Solution: True. Since $P(|X_n| < 1/2) = 0$, we see that $P(|W_n| < 1/2k) = 0$ for $n > k$.

(1 mark) (c) The random variable $Y_n = (\sum_{i=1}^{n} X_i)/n$ converges to 0 in probability.

Solution: True. Since $E(X_n) = 0$ and $\sigma^2(X) = 1/2$, we have Y_n converges to 0 in probability by the Law of Large Numbers.

(2 marks) (d) The random variable $Z_n = (\sum_{i=1}^{n} X_i)$ converges to 0 in probability.

Solution: False. By the Central Limit theorem the random variable $\sqrt{n}Y_n = Z_n/\sqrt{n}$ converges in distribution to the normal distribution. It follows that,

$$P(|Z_n| > \sqrt{n}) = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-s^2/2} ds > 0$$

Is a *fixed positive constant* for all n.

(1 mark) 4. (a) For what value of a can the following be the characateristic function of a random variable?

$$(1/3)\cos(t) + a\sin(t)/t$$

(a) **2/3**

(2 marks) (b) The characteristic function of a random variable X is given by

$$(1/3)\cos(t) + (1/3)\cos(2t) + (1/3)\exp(3t\sqrt{-1})$$

Calculate $E(X)$ and $\sigma^2(X)$ for this random variable.

Solution: We calculate the derivative at $t = 0$ to get $\sqrt{-1}$. This means that $E(X) = 1$.

We calculate the second derivative at $t = 0$ to get

$$-(1/3) - (1/3)4 - (1/3)(9)$$

This gives $E(X^2) = 14/3$. Thus $\sigma^2(X) = E(X^2) - E(X)^2 = 11/3$.