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The Effect of a Lumpy Matter Distribution on the Growth of Irregularities in an Expanding Universe*

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Summary. Given that matter is distributed in a lumpy fashion, in galaxies and clusters of galaxies in the present universe, perhaps in small lumps at an early epoch, one would suppose that the grainy distribution has some effect on the growth of irregularities. It has been suggested that there is a substantial minimum rate of growth of structure once grains form. However, it is shown here that the effect has been overestimated, so that it seems questionable whether it could have played an interesting rôle in the origin of galaxies.

Key words: cosmology – galaxies (origin)

I. Introduction

a) The Question

The question to be discussed is, what is the effect of a lumpy distribution of matter on the growth of irregularities in an expanding universe? To keep the disussion simple, it will be supposed that the universe can be approximated by the Einstein-de Sitter cosmological model ($q_0 = 1/2$, $\Lambda = p = 0$), that the typical distance between grains and the sizes of any grain clusters are much smaller than the horizon ct, and that non-gravitational forces may be ignored. Then it has been suggested (in two contexts, by Carlitz *et al.*, 1973, and by Press and Schechter, 1974) that there is a *minimum rate* at which irregularities develop out of the original granular distribution, such that

$$M(t) \propto t\,. \tag{1}$$

Here $M(t)$ is the typical mass of the gravitationally bound lumps of matter that are fragmenting out of the general expansion at epoch t.

This is an important result if true, for, as pointed out by Press and Schechter, it would strongly sharpen the predictions of the gravitational instability picture. One notices also that the mass within the horizon varies in proportion to t, the same as the minimum rate of growth of irregularities fixed by Eq. (1). Thus, following Carlitz *et al.*, one can imagine that in the very early universe some local process (perhaps involving the strong or electromagnetic interactions) causes an initially smooth and homogeneous distribution of matter to break up into small grains. Causality would suggest that the grains must be non-relativistic, grain size $< ct$. As the universe expands the grains would grow, with $M(t)$ a constant fraction of the horizon, and in time would reach the mass of a galaxy. The argument as stated here ignores the complexities introduced by the Primeval Fireball radiation, but it does illuminate the proposed principle: that causal processes operating on microscopic scales in the very early universe could set in train a gravitational process that leads to the development of irregularities on the scale of the galaxies.

My purpose here is to argue that, while there is a minimum rate of increase of $M(t)$ under the assumed conditions, it is fixed by the equation

$$M(t) \propto t^{4/7}\,. \tag{2}$$

This says that the ratio of $M(t)$ to the mass in the horizon varies as $t^{-3/7}$. Thus the development of large-scale structure in the present universe could not have been initiated by causal processes operating at epoch t if t were too small. Of course this does not vitiate the gravitational instability picture, for by adjusting the spectrum of initial irregularities one can always increase the rate of change of $M(t)$. It does say that, until we can find a deeper theory of the nature of the universe, we cannot hope to *predict* the existence of galaxies from our knowledge of (or speculations on) microscopic processes in the very early universe. However, the growth rate given by Eq. (2) still might be interesting. For example, we might suppose that the present value of M is $10^{16}\,M_\odot$, perhaps the mass of a supercluster of galaxies. Then in the Einstein-de Sitter model $M(t)$ would have been equal to the mass within radius ct at redshift $z \sim 10^8$, when the matter density would have been $\sim 10^{-5}\,\mathrm{g\,cm^{-3}}$. This is the earliest epoch at which some causal process

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could have caused an initially smooth mass distribution to fragment into a granular distribution on the desired scale. It is hard to see how the strong or electromagnetic interactions could have forced the development of lumps of size $\sim ct$ when the matter density is this low, although, given our meager understanding of what the universe may have been like at such a large redshift, it would be dangerous to conclude that we can rule it out.

b) Comparison of the Two Arguments

The two estimates of $M(t)$ are based on different measures of irregularity on the scale x_0: the power spectrum of the mass density function at $k_0=2\pi x_0^{-1}$, and the variance $(\delta M)^2$ of the mass within a randomly placed sphere of radius x_0. The discrepancy arises because the latter quantity can "see" components of the spectrum at wave number $k \gg x_0^{-1}$ (wavelength $\ll x_0$) through the sidelobes of the window function fixed by the sphere. If the power spectrum increases with increasing k more rapidly than the first power of k then the dominant contribution to $(\delta M)^2$ is from the power spectrum at wavelengths much less than x_0.

Equation (2) is based on the linear perturbation result. One introduces the fractional density contrast and its Fourier transform,

$$\delta(\boldsymbol{x},t)=(\varrho(\boldsymbol{x},t)-\varrho_0(t))/\varrho_0(t),\qquad \delta_{\boldsymbol{k}}(t)=\int_V \delta(\boldsymbol{x},t)\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\,d^3x/V. \tag{3}$$

Here $\varrho_0(t)$ is the mean density at epoch t. The expanding spatial coordinates $\boldsymbol{x}$ are related to the usual Cartesian coordinates $\boldsymbol{r}$ by

$$\boldsymbol{x}=\boldsymbol{r}/a. \tag{4}$$

The integral in Eq. (3) is over the fixed volume V in $\boldsymbol{x}$ coordinates. I take it that $\delta(\boldsymbol{x},t)$ is periodic in V. This is only a mathematical convenience, not a physical assumption, for V is much larger than the size of any clustering in the matter distribution. The expansion parameter a satisfies the cosmological equation

$$\frac{d^2a}{dt^2}=-\frac{4\pi}{3}G\varrho_0 a. \tag{5}$$

In linear perturbation theory (and for the Einstein-de Sitter cosmological model) $\delta_{\boldsymbol{k}_0}(t)$ is a sum of a growing term $\propto t^{2/3}$ and a decaying term $\propto t^{-1}$. When is this linear approximation valid? A standard criterion is based on the contribution to the variance of $\delta(\boldsymbol{x})$ by Fourier components with wave number $\leqq k_0$,

$$\langle\delta^2(x_0)\rangle \equiv \sum_{k\leqq k_0}|\delta_{\boldsymbol{k}}|^2. \tag{6}$$

If this number is less than unity, then, it is argued, the density contrast on the scale of x_0 is small and the linear perturbation result should be valid. When $\langle\delta^2(x_0)\rangle$ approaches unity the density contrast on the scale x_0 approaches unity, and bound systems of this size should separate out from the general expansion. If the power spectrum at long wavelength is a power law,

$$|\delta_{\boldsymbol{k}}|^2 \propto k^n, \tag{7}$$

this argument says that the typical mass of systems that fragment out of the general expansion at epoch t varies with time as

$$M(t)\propto t^{4/(n+3)}. \tag{8}$$

Can we place any restriction on n? Suppose that, starting with a constant mass density, we draw the material up into small lumps. By shifting each mass element through a small distance we in effect differentiate the density function, and hence introduce a factor k in the Fourier transform $\delta_{\boldsymbol{k}}$. Under the assumption that there is no short-range order among the positions of the lumps, we then conclude that the power spectrum varies as k^2 at long wavelength. However, this operation ignores momentum conservation. Suppose the lumps were formed by some local non-gravitational process that conserves momentum (or by the gravitational instability operating on earlier, smaller, lumps). Then the shifts of mass elements cancel in pairs, leaving second derivatives in the density function, k^2 in the Fourier transform, or k^4 in the power spectrum. Thus it appears that $n\leqq 4$. This limit in Eq. (8) yields Eq. (2).

Equation (1) is based on an energy argument. One considers the frequency distribution of $M(\boldsymbol{x})$, the mass within a randomly placed sphere of radius x_0 (measured in the expanding $\boldsymbol{x}$ coordinates). Let M and $(\delta M)^2$ be the mean and variance of $M(\boldsymbol{x})$. Then it is assumed that $GM\,\delta M/(ax_0)$ is a rough measure of the scatter of gravitational energy of irregularities on the scale of x_0, so that this quantity is the typical binding energy of developing proto-lumps of size $\sim x_0$, mass M. Knowing the binding energy one can estimate when the proto-system will stop expanding. The result is expressed by the equation

$$\frac{\delta M}{M}\left(\frac{t}{t_i}\right)^{2/3}\sim 1. \tag{9}$$

Here δM is evaluated at the starting time t_i and the lumps fragment out at time t. This argument certainly is valid for spherically symmetric irregularities (cf. e.g., Peebles, 1969, pp. 25–26), but it is perhaps less convincing when matter is distributed in lumps. Now δM ought to vary with M at least as rapidly as

$$\delta M \propto M^{1/3}. \tag{10}$$

As in the discussion of the power spectrum, one imagines starting with a uniform density, $\varrho=$ constant, and drawing the matter in separate non-overlapping patches into separate lumps. Then δM is fixed by the mass drawn across the surface $|\boldsymbol{x}'-\boldsymbol{x}|=x_0$. If there is not a short-range correlation among the positions of the lumps then $(\delta M)^2$ is proportional to the surface area $\propto x_0^2 \propto M^{2/3}$. This yields Eq. (10), and Eqs. (9) and (10) give Eq. (1). Equation (10) does not depend on the momentum conservation condition, because it is only a question of how much mass moves across the surface of x_0, not how the material immediately within the surface may be rearranged.

Now it is easy to see how the discrepancy between Eqs. (1) and (2) comes about. Since $M(\boldsymbol{x})$ is the result of an integral over the density function, one can express $(\delta M)^2$ in terms of the transform of the density,

$$\left(\frac{\delta M}{M}\right)^2 = \sum_{\boldsymbol{k}} |\delta_{\boldsymbol{k}}|^2 \, W(k x_0) = \frac{V}{(2\pi)^3} \int d^3 k \, |\delta_{\boldsymbol{k}}|^2 \, W(k x_0) \,, \qquad W(y) = \frac{3}{y^6} (\sin y - y \cos y)^2 \,. \tag{11}$$

Since $M(\boldsymbol{x})$ is the convolution of $\varrho(\boldsymbol{x})$ with a window function (equal to unity for argument $|\boldsymbol{x}| \leqq x_0$, equal to zero for $|\boldsymbol{x}| > x_0$), the spectrum of $M(\boldsymbol{x})$ is the product of the spectra of the two functions. The power spectrum of the window function, $W(k x_0)$, is nearly equal to unity for $k x_0 < 1$, and has side-lobes at $k x_0 > 1$ that fall off as $(k x_0)^{-4}$. If the power spectrum $|\delta_k|^2$ does not increase too rapidly with increasing k, then the sum in the first of Eq. (11) effectively is cut off at $k_0 \sim x_0^{-1}$, and we have

$$(\delta M/M)^2 \simeq \langle \delta^2(x_0) \rangle \,, \tag{12}$$

where the right hand side is defined by Eq. (6). In this case the two measures of the irregularity on scale x_0 are nearly equivalent and the two arguments give the same $M(t)$. However, suppose $n > 1$ [Eq. (7)]. Then in Eq. (11) *the dominant part of the sum is in the sidelobes.* The value of $\delta M/M$ is determined by the power spectrum where it breaks away from the power law behavior of Eq. (7), not by the spectrum at $k \sim x_0^{-1}$, and of course $\delta M/M$ is much larger than $\langle \delta^2(x_0) \rangle$. One notices also from Eq. (11) that in this case $(\delta M/M)^2 \propto x_0^{-4} \propto M^{-4/3}$, in agreement with Eq. (10).

c) Lines of Attack

Now the problem is to decide which if either of the two approaches may be trusted. For the perturbation theory argument, one question is clear. When matter is distributed in a grainy fashion linear perturbation theory certainly is not adequate to describe the small-scale behavior of the distribution. Under what conditions can we apply linear perturbation theory to the time dependence of the Fourier components $\delta_{\boldsymbol{k}}$? In the next section the general equation for the time variation of $\delta_{\boldsymbol{k}}$ is written down. This equation looks like the linear perturbation theory result with the addition of some terms that, under reasonable circumstances, vary as k^4. Thus, if $n < 4$ in the original power spectrum [Eq. (7)], linear perturbation theory applies for long enough wavelengths. This shows that the energy argument is literally false, for if $\delta M/M$ measured the energy of proto-lumps then the evolution of irregularities associated with the lumps would fill in the power spectrum at small k to give $n = 1$.

Discussion of the limits of perturbation theory does not address the spirit of the "self-similar gravitational bootstrap" conjecture discussed by Press and Schechter. Starting with a grainy distribution of matter, they observe that the time required for neighboring grains to fall together and form larger grains is comparable to the expansion time-scale. The picture is that the growth of irregularities in the expanding universe might be entirely due to this non-linear interaction among neighboring newly-formed lumps, and the speculation is that the rate of growth of structure in this non-linear process is fixed by Eq. (1). This is a very difficult concept to analyze because the process falls outside the domains of perturbation theory on the one side and equilibrium arguments on the other. One possible point of attack is to make use of a remarkable energy equation discovered by Layzer (1963; cf. Irvine, 1965). This equation relates the mean square peculiar matter velocity (relative to the uniform general expansion) to an integral over the power spectrum of the density irregularities. The only assumptions are that the spatial distribution of matter is statistically homogeneous and that the Newtonian approximation is valid. In § III it is shown that if the bootstrap conjecture were valid the mean square peculiar velocity of matter would vary with n [Eq. (7)] in a way opposite to what seems reasonable on other grounds.

II. Perturbation Theory and the Fine-grain Average

a) The General Equations

I assume that matter is concentrated in point masses m_i at coordinates $\boldsymbol{x}_i(t)$. The particles are distributed through space in a statistically uniform way (stationary isotropic random process), and the distribution is expanding roughly in accordance with Hubble's law. I assume the particles are so light and numerous that one can choose a volume V such that (1) the number N of particles in V is large; (2) structure in the distribution of points on the scale V is negligibly small; and (3) the expansion within V is non-relativistic.

For the point-particle distribution, Eq. (3) becomes (for $k \neq 0$)

$$\delta_{\boldsymbol{k}}(t) = \sum_j m_j \exp(i \boldsymbol{k} \cdot \boldsymbol{x}_j)/M_V \,, \tag{13}$$

where M_V is the total mass within V,

$$M_V = \varrho_0 a^3 V \,. \tag{14}$$

The factor a^3 comes from the change of variables from $\boldsymbol{x}$ to $\boldsymbol{r}$ [Eq. (4)]. Now consider the second time derivative of δ_k as defined by Eq. (13). This quantity depends on first and second derivatives of $x_j(t)$. For the latter we can use Newton's equations of motion,

$$\frac{d^2 r_j}{dt^2} = a\frac{d^2 x_j}{dt^2} + 2\frac{da}{dt}\frac{dx_j}{dt} + x_j\frac{d^2 a}{dt^2} = -\frac{\nabla\varphi_j(x_j)}{a},$$

$$\varphi_j(x) = \tfrac{2}{3}\pi G\varrho_0 a^2 x^2 + \psi_j(x), \tag{15}$$

$$\frac{\nabla^2\psi_j(x)}{a^2} = 4\pi G\left[\sum_{l\neq j}\frac{m_l\,\delta(x-x_l)}{a^3} - \varrho_0\right].$$

The first equation expresses the acceleration in terms of the x coordinates [Eq. (4)]. The potential is written as the sum of two terms, so that the source for the second term has zero spatial average and can be expressed as a Fourier series,

$$\psi_j(x) = -\frac{4\pi G}{aV}\sum_{k\neq 0}\sum_{l\neq j}\frac{m_l}{k^2}\exp ik\cdot(x-x_l). \tag{16}$$

The result of using Eqs. (15) and (16) in the expression for the second derivative of δ_k is

$$\frac{d^2\delta_k}{dt^2} + \frac{2}{a}\frac{da}{dt}\frac{d\delta_k}{dt} = 4\pi G\varrho_0\delta_k + A - C, \tag{17}$$

where A and C are defined by the equations

$$A = 4\pi G\varrho_0\sum_{k'\neq 0,k}\frac{k\cdot k'}{k'^2}B(k'),$$

$$B(k') = \frac{1}{M_V^2}\sum_{j\neq l} m_j m_l \exp i[(k-k')\cdot x_j + k'\cdot x_l] \tag{18}$$

$$\simeq \delta_{k-k'}\delta_{k'},$$

$$C = \sum_j \frac{m_j}{M_V}\left(k\cdot\frac{dx_j}{dt}\right)^2 \exp ik\cdot x_j.$$

In the equation that defines A the component $k'=0$ is not included in the sum because it does not appear in Eq. (16), and the component $k'=k$ is written separately because it is much larger then the other components. This gives the first term on the right hand side of Eq. (17), where I have used the approximation

$$M_V\delta_k \gg \sum m_j^2 \exp ik\cdot x_j,$$

valid when $N\gg 1$. It is convenient finally to rewrite the equation for A by replacing the index of summation with $k''=k-k'$. This gives

$$A = 2\pi G\varrho_0\sum_{k'\neq 0,k} D(k,k')\,B(k'),$$

$$D = \frac{k\cdot k'}{k'^2} - \frac{k\cdot(k'-k)}{|k'-k|^2}. \tag{19}$$

Equation (17) is the usual zero-pressure first-order perturbation theory result, with the addition of two terms A and C that can represent non-linear effects. (The term C can also represent the effect of kinetic gas pressure.) Thus to find a criterion for the validity of perturbation theory we must estimate the magnitudes of A and C.

b) Discussion

As a first step, let us suppose that at some chosen starting time the particles have no peculiar velocities ($C=0$) and that the power spectrum vanishes at $k<k_1$. In the linear perturbation solution the long-wavelength part of δ_k would stay equal to zero. Equation (17) says that δ_k at $k<k_1$ starts to grow because of the source term A. Since B vanishes at $k'<k_1$ [with the exception of the very small correction term in the third of Eq. (18)], Eq. (19) say that the magnitude of this source term is approximately proportional to k^2 (for $D\propto k^2$ when $k\ll k'$). Thus at $k\ll k_1$ the Fourier spectrum grows a "tail" $\propto k^2$, the power spectrum a tail $\propto k^4$. As was remarked in § I, *because of momentum conservation this k^4 tail is the expected consequence of the small-scale motions of the particles.*

Let us consider next the expected magnitude of A under some simple assumptions. If the points x_j are distributed at random in V then

$$\langle|\delta_k|^2\rangle = N\langle m^2\rangle/M^2 \simeq N^{-1}. \tag{20}$$

To estimate $\langle|A|^2\rangle$ we note that, when the particles are randomly distributed, $\langle B(k')B^*(k'')\rangle$ vanishes unless $k''=k'$ or $k''=k-k'$, and in either of these cases the mean value is

$$\langle B(k')B^*(k'')\rangle \simeq N^2\langle m^2\rangle^2 M_V^{-4} \simeq N^{-2}. \tag{21}$$

Then on changing the sum over k' to an integral, we have from Eq. (19)

$$\langle|A|^2\rangle \simeq \frac{1}{2}(4\pi G\varrho_0)^2\frac{V}{(2\pi)^3}\int d^3k'\,D^2/N^2. \tag{22}$$

Since

$$D^2 \sim (k/k')^2, \quad k'<k, \qquad \sim (k/k')^4, \quad k'>k, \tag{23}$$

the integral converges for small and large k', giving

$$\langle|A|^2\rangle \sim (G\varrho_0)^2\,Vk^3/N^2. \tag{24}$$

Comparing this result with Eq. (20), we see that A may be neglected compared to $G\varrho_0\delta_k$ *if there are many points per cubic wavelength.*

Next, let us consider the "sub-random" distribution, where the δ_k have random phases and the power spectrum is approximately

$$|\delta_k|^2 = N^{-1}, \quad k>k_1; \qquad |\delta_k|^2 = N^{-1}(k/k_1)^n, \quad k<k_1; \qquad k_1 = 2\pi(N/V)^{1/3}. \tag{25}$$

This describes a point distribution that "looks" random on scales less than or comparable to the mean distance between points – there is no short-range order – but that looks smoother than random on larger scales. Since the δ_k are assumed to have random phases we can use the second of the equations for B in Eq. (18)

with Eq. (25) to find the expected value of $B(\boldsymbol{k}')\,B^*(\boldsymbol{k}'')$. The result is that if $n>1/2$ and $k \ll k_1$ the dominant contribution to $\langle |A|^2 \rangle$ is from $k' \sim k_1$ and amounts to

$$\langle |A|^2 \rangle \sim (G\varrho_0)^2\,(k/k_1)^4\,N^{-1}\,. \tag{26}$$

Comparing this with Eq. (25), we see that A may be neglected compared to $G\varrho_0\,\delta_{\boldsymbol{k}}$ if $n<4$ and $k \ll k_1$, again in agreement with the argument of § I.

Now let us consider the role of the term C. We are assuming that the peculiar velocities vanish at the starting time. After one expansion time $\sim (G\varrho_0)^{-1/2}$ the gravitational interaction among neighboring particles builds up a velocity dispersion

$$v^2 = a^2 \left(\frac{d\boldsymbol{x}}{dt}\right)^2 \sim Gmk_1/a\,, \tag{27}$$

where ak_1^{-1} is the typical particle separation [Eq. (25)]. If the particles are randomly distributed we can assume that the last of Eq. (18) is a sum over uncorrelated terms, so the mean of the square of C is

$$\langle |C|^2 \rangle \simeq \frac{k^4 N m^2 v^4}{M_V^2 a^4} \sim (G\varrho_0)^2\,(k/k_1)^4\,N^{-1}\,. \tag{28}$$

If the distribution were sub-random [in the sense of Eq. (25)] it would not necessarily mean that the terms in C are distributed in a sub-random fashion, but I expect that $|C|^2$ would not be greater than the random case [Eq. (28)]. Thus, after one expansion time C should not exceed A [Eq. (26)].

What happens to C as the universe expands through several more expansion times? Gravitationally bound clusters of particles form, and the velocities of particles within clusters can make a large contribution to C. However, we know that this contribution must cancel some of the terms in A, because we can re-do the analysis treating each bound cluster as a new "point particle". As may be verified, this conclusion also follows from the virial theorem.

I conclude that when the long wavelength part of the spectrum is a power law with $n<4$ [Eq. (25)], the usual zero-pressure first-order perturbation result is a valid approximation for Fourier components with wavelength much greater than the inter-particle distance. This might be compared to the energy argument. Let us start again with the particle distribution characterized by Eq. (25). Then $\delta M/M$ is given by Eq. (11) and satisfies Eq. (10) if $n>1$. The argument is that in the original particle distribution there are proto-clusters of mass M and radius $R\,(M \sim \varrho_0 R^3)$ with negative energy (gravitational plus potential) amounting to $\sim GM\,\delta M/R$. Judging from the evolution of spherically symmetric systems, the evolution of one of these proto-clusters can be expressed as the sum of a growing mode of perturbation and a decaying one, where the growing mode is constructed from a uniform mass distribution by shifting matter by the amount $\delta R \sim R\delta M/M$ on the scale R. After several expansion times, when the growing mode dominates, the actual perturbation to particle positions would be comparable to this value for δR. But if the shifts of particle positions were this large, the effect would show up in the long-wavelength part of the transform of the distribution as $|\delta_{\boldsymbol{k}}|^2 \propto k$. Since it has been shown that this does not happen, the energy argument is false when $n>1$.

III. The Layzer-Irvine Equation

The following discussion is based on the point particle picture and on the Newtonian approximation used in § II. The basic equation in the form that will be needed here is derived in part (a) [Eq. (32), below]. The equation is used in a discussion of the bootstrap conjecture in part (b).

a) Derivation

The computation proceeds along the lines of the usual derivation of energy conservation in Newtonian mechanics, except that the expanding coordinates of Eq. (4) are used. On multiplying the second of Eq. (15) by $am_j\,d\boldsymbol{x}_j/dt$, summing over all particles j in V, using Eq. (5), and re-arranging some terms, one finds

$$\begin{aligned}&\left(\frac{d}{dt} + \frac{2}{a}\frac{da}{dt}\right) \sum_j \frac{1}{2}\,m_j a^2 (d\boldsymbol{x}_j/dt)^2 \\ &= \frac{G}{2a}\,\frac{d}{dt} \sum_j m_j \left(\sum_{l \neq j} \frac{m_l}{|\boldsymbol{x}_j - \boldsymbol{x}_l|} + \frac{4}{3}\pi\varrho_0 a^3 \boldsymbol{x}_j^2\right).\end{aligned} \tag{29}$$

This is the usual Newtonian energy conservation law. The Layzer-Irvine energy equation follows on setting

$$K = \sum_j \tfrac{1}{2} m_j a^2 (d\boldsymbol{x}_j/dt)^2/M_V\,, \tag{30}$$

and subtracting some constant terms from the quantity in the parentheses on the right hand side of Eq. (29).

Layzer pointed out that the right hand side of Eq. (29) can be expressed in terms of the covariance function of the mass density (considered as a continuous function). This is readily carried over to the point particle picture. With $\delta_{\boldsymbol{k}}$ defined in Eq. (13), we have

$$\begin{aligned}\int \frac{d^3k}{k^2}\,(|\delta_{\boldsymbol{k}}|^2 - \gamma) &= \frac{1}{M_V^2} \sum_{j \neq l} m_j m_l \int \frac{d^3k}{k^2} \exp i\boldsymbol{k}\cdot(\boldsymbol{x}_j - \boldsymbol{x}_l) \\ &= \frac{2\pi^2}{M_V^2} \sum_{j \neq l} \frac{m_j m_l}{|\boldsymbol{x}_j - \boldsymbol{x}_l|}\,, \\ \gamma = \sum m_j^2/M_V^2 \simeq N^{-1}\,.\end{aligned} \tag{31}$$

Using this result in Eq. (29), and observing that the last term in parentheses on the right hand side is very nearly a constant which can be dropped, we arrive at the desired energy equation,

$$\begin{aligned}&\frac{d}{dt}(K+U) + \frac{1}{a}\frac{da}{dt}(2K+U) = 0\,, \\ &U = -\frac{GM_V}{\pi a} \int_0^\infty dk\,(|\delta_{\boldsymbol{k}}|^2 - \gamma)\,.\end{aligned} \tag{32}$$

In the second equation one imagines replacing the power spectrum with its ensemble average, which is assumed to be a function of the magnitude of $\boldsymbol{k}$, so the integral over the direction of $\boldsymbol{k}$ is directly evaluated. Since the power spectrum is the Fourier transform of the covariance function, the equation for U can be rewritten as an integral over the covariance function, which agrees with Layzer's result [Layzer, 1963, Eq. (28), after correction for a missing factor of two in this equation].

b) Discussion

The time rate of change of the "cosmic kinetic energy" K is a measure of the rate of growth of irregularities, and Eq. (32) gives a relation between this quantity and the power spectrum. Thus we can hope to gain some understanding of how the long wavelength part of the spectrum affects the growth of irregularities.

The first step is to obtain the time dependence of K and U when Eq. (1) is valid. In this case the typical size $l(t)$ of the systems that separate out from the general expansion at epoch t varies as $l \propto t$ because the mean density varies as $\varrho_0 \propto t^{-2}$. In one characteristic expansion time t the growth of irregularities adds to K (and subtracts from U) an amount $\sim GM(t)/l(t)$, which is independent of t. We have then

$$\frac{dK}{dt} = \frac{\alpha}{t}, \qquad \alpha \cong \frac{GM(t)}{l(t)}, \tag{33}$$

where α is a constant. The first of Eqs. (32) and (33) have the general solution

$$\begin{aligned} K &= \alpha \log t/t_0 , \\ U &= \tfrac{3}{2}\alpha - 2\alpha \log t/t_0 + \beta t^{-2/3} . \end{aligned} \tag{34}$$

The constant t_0 is a measure of the epoch at which the first bound systems in the hierarchy form. The constant β reflects the freedom of choice of zero level for the potential.

The next step is to compare the evolution of two model universes. Both commence with the matter distributed in grains of the same size. In one model the long wavelength part of the spectrum has $n=1$ [Eq. (7)]. Then Eqs. (1) and (34) follow by either argument. In the second model the long wavelength part has $n>1$. By the bootstrap conjecture bound systems are forming at the same rate in the two models so the power spectra at short wavelength are the same. Since we have already established that the long wavelength part of the spectrum is growing less rapidly in the second model than in the first (because it started out smaller), the spectrum in the second model must steepen at long wavelength, approaching the limit $n=4$. Thus it is sufficient to compare two models, with $n=1$ and $n=4$.

To estimate the spectra in the models we note that at epoch t irregularities on the scale $l(t)$ are going nonlinear. Thus we have

$$|\delta_{k_l}|^2 \simeq M(t)/M_V , \qquad k_l = 2\pi\, a(t)/l(t) . \tag{35}$$

By the conjecture the spectra are the same for $k \gtrsim k_l$, so

$$|\delta_{\boldsymbol{k}}|^2 \simeq \frac{M(t)}{M_V}\left(\frac{k}{k_l}\right)^n , \quad n=1,4; \quad k<k_l . \tag{36}$$

The power spectra can be used to compare the covariance functions $\xi(x)$ and the potentials U. For the covariance function, I assume for simplicity that all particles have the same mass m. Then ξ is defined in terms of the probability δP that a particle is found in the volume element δV at distance x in a randomly chosen direction from a randomly chosen particle in V,

$$\delta P = \frac{N}{V}\,\delta V(1+\xi(x)) . \tag{37}$$

This function measures the mean density at distance x from a particle, and it can be considered the analog of the autocovariance function (lagged product) for a continuous density function. It is related to the power spectrum by the equation [Peebles, 1973, Eqs. (1), (16)]

$$\xi(x) = \frac{V}{2\pi^2}\int_0^\infty k^2\, dk\, \frac{\sin kx}{kx}\,(|\delta_{\boldsymbol{k}}|^2 - N^{-1}) . \tag{38}$$

For the assumed power spectra ξ at $x \sim l(t)/a(t)$ is smaller for the $n=4$ model than for the $n=1$ model because the long wavelength part of the spectrum gives a smaller contribution to the integral. As measured by ξ, the large-scale matter distribution is *smoother* for the $n=4$ model. For the potential energy, Eqs. (32) and (36) give

$$U_4 - U_1 \simeq \frac{3}{5}\,\frac{GM(t)}{l(t)} = \varepsilon\alpha , \tag{39}$$

where $\varepsilon \simeq 1$ and the constant α is defined in Eq. (33). The magnitude of the potential U is smaller for the model with $n=4$, in agreement with the smaller value for $\xi(l/a)$.

Both models are supposed to satisfy Eq. (34), which can be consistent with Eq. (39) if the t_0 values are in the right ratio. But this gives

$$K_4 - K_1 = -\varepsilon\alpha/2 . \tag{40}$$

That is, the velocity dispersion in the model with $n=4$ is smaller than for $n=1$ by an amount comparable to the increment in the dispersion in one expansion time.

The final step is to argue that we really would have expected $K_4 > K_1$.

The mass distribution starts out nearly the same in the two models, close to uniform, and is supposed to end up in similar-looking lumps. Since the motion from initial to final state is more gradual in the $n=1$ model, we would expect that the mean square velocity is smaller. To make this more explicit, let us write

$$K_n = \int_{\mu_0}^{\infty} \frac{d\mu}{\mu} v_n^2(\mu, t), \quad (41)$$

where v^2 is the contribution to K by lumps (or proto-lumps) of mass $\sim\mu$. Equation (1) says that mass scales with time, so we have the scaling relation

$$v_n^2(\mu, t) = f_n(t/\mu), \quad (42)$$

with the limiting values

$$\begin{aligned} f_n(x) &\simeq \alpha, \quad x \gg t/M(t), \\ &\simeq 0, \quad x \ll t/M(t). \end{aligned} \quad (43)$$

The first limit says that the velocity dispersion within a lump is independent of when the lump formed [Eq. (33)]. The lower bound on the integral in Eq. (41) is the mass of the first bound systems to have formed. When $M(t) \gg \mu_0$, Eqs. (41)–(43) say K varies like $\alpha \log t$, in agreement with Eq. (34).

The difference of kinetic energies in the models is

$$K_4(t) - K_1(t) \simeq \int_{M(t)}^{\infty} \frac{d\mu}{\mu} [f_4(t/\mu) - f_1(t/\mu)]. \quad (44)$$

Since it is assumed that bound systems are similar in the two models the integral is over the developing irregularities only. With the variable change $t' = M(t)\, t/\mu$, Eq. (44) becomes

$$K_4(t) - K_1(t) = \int_0^t \frac{dt'}{t'} [v_4^2(M(t), t') - v_1^2(M(t), t')]. \quad (45)$$

This is the difference of integrals over the velocity dispersion at a *fixed* mass $M(t)$ in the two models. We can approximate these integrals as

$$I = \int_0^t \frac{dt'}{t'} a(t')^2 \left(\frac{dx(t')}{dt'}\right)^2, \quad (46)$$

where $x(t')$ measures the motion (in one of the components x^α) of the matter on the scale $M(t)$ as it evolves from the uniform to the final (lumpy) state. The minimum value of I subject to the constraints that $x(t)$ commence and end up at fixed values, is obtained when

$$x(t) = A + B t^{2/3}, \quad (47)$$

where A and B are constants. But this is just the time variation of $x(t')$ in linear perturbation theory (for $\delta\varrho/\varrho \propto t^{2/3}$). Since Model 1 is supposed to approximate linear perturbation theory, while in Model 2 the motion of developing irregularities is supposed to be slower at first, more rapid at $t' \sim t$, it follows that $K_4 > K_1$. This contradicts Eq. (40).

The argument may be summarized as follows. When the rate of growth of irregularities is given by Eq. (1), the peculiar velocity dispersion is given by Eq. (33) and the Layzer-Irvine equation fixes the potential energy [Eq. (34)]. Equation (1) follows from the linear perturbation argument if the spectrum of density irregularities at long wavelength varies as $|\delta_k|^2 \propto k$. In the bootstrap conjecture, it is argued that the equation may be valid independent of the spectrum at long wavelength. But then it would follow that, if we reduced this function below $|\delta_k|^2 \propto k$, we would reduce the magnitude of the potential U [Eqs. (32), (39)], and hence reduce the velocity dispersion. This contradicts the argument that the dispersion is minimum when irregularities develop according to the linear picture rather than the bootstrap picture [Eq. (47)].

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