Solutions to Assignment 9

1. Check the following summations as a way of verifying the formulas for characteristic functions of some discrete probability distributions.

   (a) For the Binomial distribution with $p+q=1$

$$\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}\exp(kt\sqrt{-1}) = (q+pe^{t\sqrt{-1}})^n$$

> **Solution:** Using the fact that $\exp(kt\sqrt{-1}) = \exp(t\sqrt{-1})^k$, we have
>
> $$\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}\exp(kt\sqrt{-1}) = \sum_{k=0}^{n}\binom{n}{k}\left(p\exp(t\sqrt{-1})\right)^k q^{n-k}$$
>
> Now the Binomial theorem gives the result.

   (b) For the Poisson distribution

$$\sum_{k=0}^{\infty}\frac{c^k}{k!}\exp(-c+kt\sqrt{-1}) = \exp\left(c\left(e^{t\sqrt{-1}}-1\right)\right)$$

> **Solution:** We can pull out $\exp(-c)$ and also use the fact that $\exp(kt\sqrt{-1}) = \exp(t\sqrt{-1})^k$, to write
>
> $$\sum_{k=0}^{\infty}\frac{c^k}{k!}\exp(-c+kt\sqrt{-1}) = \exp(-c)\sum_{k=0}^{\infty}\frac{(ce^{t\sqrt{-1}})^k}{k!}$$
>
> Now, the exponential series gives
>
> $$\exp(-c)\sum_{k=0}^{\infty}\frac{(ce^{t\sqrt{-1}})^k}{k!} = \exp(-c)\exp(ce^{t\sqrt{-1}})$$
>
> from which we get the formula as desired.

   (c) For the Negative Binomial distribution

$$\sum_{k=0}^{\infty}\binom{n+k-1}{k}p^n q^k\exp(kt\sqrt{-1}) = \left(\frac{p}{1-qe^{t\sqrt{-1}}}\right)^n$$

Solution: Using the fact that $\exp(kt\sqrt{-1}) = \exp(t\sqrt{-1})^k$, and pulling out p^n, we have

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} p^n q^k \exp(kt\sqrt{-1}) = p^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} (qe^{t\sqrt{-1}})^k$$

Now the result follows from the Negative Binomial expansion or the Taylor series for $1/(1-T)^n$.

2. Check the following integrals as a way of verifying the formulas for characteristic functions of some probability densities.

   (a) For the uniform density

$$\frac{1}{2}\int_{-1}^{1} \exp(at\sqrt{-1})da = \frac{\sin(t)}{t}$$

   Justify the limits of the integral.

Solution: Let $u(a)$ denote the uniform density which is $1/2$ between -1 and 1 and 0 outside this interval. By definition, the characteristic function of the uniform distribution is given by

$$\int_{-\infty}^{\infty} e^{at\sqrt{-1}} u(a)da$$

By the above description of $u(a)$ we simplify this to

$$\frac{1}{2}\int_{-1}^{1} e^{at\sqrt{-1}}da = \frac{1}{2}\left.\frac{e^{at\sqrt{-1}}}{t\sqrt{-1}}\right|_{a=-1}^{a=1}$$

The latter expression becomes

$$\frac{e^{t\sqrt{-1}} - e^{t\sqrt{-1}}}{2t\sqrt{-1}} = \frac{\sin(t)}{t}$$

   (b) For the Poisson density (exponential distribution)

$$\int_{0}^{\infty} c\exp(-ca + at\sqrt{-1})da = \frac{c}{c - t\sqrt{-1}}$$

   where $c > 0$.

Solution: We write the integral

$$\int_0^\infty c\exp(-ca + at\sqrt{-1})da = c\int_0^\infty \exp(a(-c + t\sqrt{-1}))da$$

The latter expression becomes

$$c\,\frac{\exp(a(-c+t\sqrt{-1}))}{-c+t\sqrt{-1}}\Big|_{a=0}^{a=\infty}$$

Since $\exp(a(-c+t\sqrt{-1}))$ goes to 0 as a goes to infinity, we see that this becomes

$$\frac{c}{c - t\sqrt{-1}}$$

as required.

(c) For the normal density

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-a^2/2 + at\sqrt{-1})da = \exp(-t^2/2)$$

Solution: First of all we check

$$\frac{d(a^n e^{-a^2/2})}{da} = (na^{n-1} - a^{n+1})e^{-a^2/2}$$

Secondly, we have

$$\int_{-\infty}^{\infty} d(a^n e^{-a^2/2}) = a^n e^{-a^2/2}\Big|_{-\infty}^{\infty} = 0$$

Finally, we also note that (substituting $b = -a$)

$$\int_{-\infty}^{\infty} ae^{-a^2/2}da = -\int_{-\infty}^{\infty} be^{-b^2/2}db$$

so that this integral is 0.
This shows that if we define A_n by

$$A_n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a^n e^{-a^2/2}da$$

Then we have $A_{n+1} = nA_{n-1}$; moreover, we have already seen that $A_0 = 1$. It follows that $A_n = 0$ for n odd. Moreover, by induction on k, we get the formula

$$A_{2k} = (2k-1)\cdot(2k-3)\cdots 1A_0 = \frac{(2k)!}{k!2^k}$$

(Note how we took care of $k = 0$ as well by multiplying and dividing by $2k!$)

Now, suppose that $f(a)$ is given by a power series $\sum_{n=0}^{\infty} c_n a^n$, *and* that we can interchange summation and integration in the integral

$$\int_{-\infty}^{\infty} f(a) e^{-a^2/2} da$$

Then, we can calculate this integral as

$$\sum_{k=0}^{\infty} c_{2k} \frac{(2k)!}{k! 2^k}$$

In our case

$$f(a) = \exp(at\sqrt{-1}) = \sum_{n=0}^{\infty} \frac{(t\sqrt{-1})^n}{n!} a^n$$

Hence, we get the integral as

$$\sum_{k=0}^{\infty} \frac{(t\sqrt{-1})^{2k}}{(2k)!} \frac{(2k)!}{k! 2^k} = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k! 2^k} = \exp(-t^2/2)$$

3. Justify the following limits either directly or using Probability theory:
   (a) $((1 - c/n) + (c/n)e^{t\sqrt{-1}})^n$ converges to $\exp(c(e^{t\sqrt{-1}} - 1))$ as n goes to infinity.

Solution: This is just the usual limit $(1 - z/n)^n$ goes to $\exp(-z)$ as n goes to infinity. We can also see it as the limit of Binomial distribution being the Poisson distribution by looking at characteristic functions.

The former distribution is the Binomial distribution with probability $p = c/n$ and we know that as n goes to infinity this converges to the Poisson distribution.

   (b) $\cos(t/\sqrt{n})^n$ goes to $\exp(-t^2/2)$ as n goes to infinity.

Solution: This is a bit tricky to prove without using Probability theory (but see below). It is a consequence of de Moivre's theorem for n unbiased coin flips (or the Random walk with n steps) as n goes to infinity.

Note that the characteristic function for random variable X of the unbiased coin flip with 1 for Head and -1 for tail is given by $\cos(t)$. Hence, if W_n denotes the sum of n independent such coin flips, then, the characteristic function for $W_n/\sqrt{n}$ is $\cos(t/\sqrt{n})^n$. By de Moivre's theorem the distribution of $W_n/\sqrt{n}$ converges to the normal distribution. Hence, the characteristic functions converge to $e^{-t^2/2}$.

(c) If c_n goes to c as n goes to infinity, then $(1-c_n/n)^n$ goes to e^{-c} as n goes to infinity.

Solution: Since log is a strictly monotone function, we can equally well try to prove that

$$n\log(1-c_n/n) \text{ goes to } -c \text{ as } n \text{ goes to } \infty$$

Now, c is a fixed constant and c_n are close to c for large n. Hence, c_n/n are less than 1/3 for large n. It follows that we can expand using the power series for $\log(1-t)$ for $|t|<1$. The result now follows from the fact that the sum of the terms from the second onwards are at most Ct^2 for $|t|<1/3$ for some positive constant C. Thus,

$$n\log(1-c_n/n) = -c_n + O(1/n)$$

for large values of n.

Returning to the previous question, we first note that the difference between $\cos(t/\sqrt{n})$ and $1-t^2/2n$ is also $o(n^2)$ for any fixed t. From this one can show that $\cos(t/\sqrt{n})^n$ is (for any fixed t) asymptotically equal to $(1-t^2/2n)^n$ as n goes to infinity. It then follows that it is asymptotically equal to $\exp(-t^2/2)$.

4. Let X be the random variable that gives the difference of the numbers appearing when two dice are rolled. Calculate the characteristic function of X.

Solution: Let $H_d(a)$ denote the distribution function which takes the value 0 for $a<d$ and 1 for $a\geq d$. We then calculate

$$F_X(a) = (2/36)H_5 + (4/36)H_4 + (6/36)H_3 + (8/36)H_2 + (10/36)H_1 + (12/36)H_0$$

Since the characteristic function of H_d is $\exp(dt\sqrt{-1})$, we see that the characteristic function for X is

$$\begin{aligned}\phi_X(a) = (2/36)\exp(5t\sqrt{-1}) + (4/36)\exp(4t\sqrt{-1}) + (6/36)\exp(3t\sqrt{-1})\\ + (8/36)\exp(2t\sqrt{-1}) + (10/36)\exp(t\sqrt{-1}) + (12/36)\end{aligned}$$