Solutions to Assignment 8

1. We roll a pair of dice, one red and one blue. The random variable X_1 denotes the value on the face of the red die and X_2 denotes the value on the face of the blue die. Which of the following random variables are independent? Justify your answer. In each case compute the covariance and correlation.
   (a) X_1 and X_2.
   (b) $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

Solution: Since there is no relation between the result on the first die and that of the second die we generally *assume* that X_1 and X_2 are independent. Thus, we have (by assumption!)

$$P(X_1 = i; X_2 = j) = P(X_1 = i) \cdot P(X_2 = j)$$

On the one hand, the event that $X_1 - X_2 = a$ and $X_1 + X_2 = b$ are equivalent to the event that $X_1 = (a + b)/2$ and $X_2 = (b - a)/2$. On the other hand the event that $X_1 - X_2 = a$ is equivalent to the *union* of the events $X_2 = i; X_1 = a + i$ for all i; hence, we can guess that these events will not be independent. Explicitly, we can see that

$$P(X_1 - X_2 = 0; X_1 + X_2 = 7) = 0 \text{ but } P(X_1 - X_2 = 0) = 1/6 \text{ and } P(X_1 + X_2 = 7) = 1/6$$

The covariance $Cov(X_1, X_2) = 0$ since these are independent. We see that

$$E((X_1 - X_2)(X_1 + X_2)) = E(X_1^2 - X_2^2) = E(X_1^2) - E(X_2^2) = 0$$

At the same time,

$$E(X_1 - X_2) = E(X_1) - E(X_2) = 0$$

Hence $Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 0 - 0 = 0$. However, these variables are *not* independent!

2. Given two random variables X and Y with $E(X^2) = 1$ and $E(Y^2) = 0.25$. Which of the following are possible and impossible?
   (a) $E(XY) = 0.25$
   (b) $E(XY) = 0.6$.
   (c) $E(XY) = 0.4$.
   (d) $E(XY) = -0.4$.

Solution: We have seen that for any constants a and b, we have

$$0 \le E((aX + bY)^2) = a^2E(X^2) + 2abE(XY) + b^2E(Y^2)$$

Using $A = E(X^2) > 0$, $B = E(XY)$ and $C = E(Y^2) > 0$, we get

$$0 \le a^2A + 2abB + c^2C = A(a + bB/A)^2 + b^2(C - B^2/A)$$

for all a and b. Putting $b = 1$ and $a = -B/A$, we see that $B^2/A - C \le 0$. Equivalently,

$$E(XY)^2 \le E(X^2)E(Y^2)$$

In particular, we can have $\{0.25, 0.4, -0.4\}$ as values for $E(XY)$, but not 0.6.

3. We repeatedly flip a fair coin. Let X_i denote the random variable that takes the value 1 if the i-th flip returns Head and 0 if it returns Tail. Which of the following statements are True? Justify your answer.

   (a) The sequence X_i converges to 1/2 almost surely.

   (b) The sequence X_i converges to 1/2 in probability.

   Solution: We have probability $P(|X_i - 1/2| > 0.25) = 1$ for all i. This does not converge to 0. Hence, X_i do not converge to 1/2 in probability. Hence, the even stronger almost sure convergence also does not hold.

   (c) The sequence X_i/i converges to 0 in probability.

   (d) The sequence X_i/i converges to 0 almost surely.

   Solution: We have $P(|X_i/i| > 1/j) = 0$ for all $i > j$. Hence, X_i/i converges to 0 almost surely.

   (e) The sequence $Y_n = \sum_{i=1}^n X_i$ converges to 1/2 in probability.

   (f) The sequence $S_n = (\sum_{i=1}^n X_i)/n$ converges to 1/2 in probability.

   (g) The sequence $S_n = (\sum_{i=1}^n X_i)/n$ converges to 1/2 almost surely.

   Solution: We have $E(X_i) = 1/2$ and $\sigma^2(X_i) = 1/4$. Moreover, the variables X_i are independent and identically distributed. Applying the Strong Law of Large numbers, it follows that the running averages $S_n = (X_1 + \cdots + X_n)/n$ converge almost surely to the mean 1/2. It follows that we also get convergence in probability. Now $Y_n = n \cdot S_n$.
   For large n we see that the event $|Y_n - 1/2| > 1$ contains the event $|Y_n/n - 1/2| \le 1$; the latter event has probility close to 1, hence $|Y_n - 1/2| > 1$ has probability

close to 1 for large n. In particular, it does not converge to 0 as n goes to infinity. Thus, Y_n does not converge to $1/2$.

(h) The sequence $U_n = \sum_{i=1}^{n} X_i/2^i$ converges to 1 in probability.

(i) The sequence $U_n = \sum_{i=1}^{n} X_i/2^i$ converges to $U = \sum_{i=1}^{\infty} X_i/2^i$ in probability.

(j) The sequence $U_n = \sum_{i=1}^{n} X_i/2^i$ converges to $U = \sum_{i=1}^{\infty} X_i/2^i$ almost surely.

Solution: We have $U - U_n = \sum_{i>n} X_i/2^i$, it follows that

$$|U - U_n| \leq \sum_{i>n} |X_i|/2^i \leq \sum_{i>n} 1/2^i = 1/2^n$$

In other words, $P(|U - U_n \leq 1/2^n) = 1$ and so $P(|U - U_n| > 1/2^m) = 0$ for $n > m$. We easily conclude that U_n converges to U almost surely.

Note that $U_n \geq 1/2$ if and only if $X_1 = 1$. Also U_n takes values between 0 and 1. Thus $|U_n - 1| < 1/2$ is the same event as $U_n \geq 1/2$. It follows that $P(|U_n - 1| < 1/2) = 1/2$ for all n. In particlar, it does not go to 0 as n goes to infinity. Hence, U_n does not converge to 1 in probability.

4. We repeatedly carry out an experiment to measure the acceleration due to gravity. Let X_i be the random variable that denotes the result of the i-th measurement. The expected value of X_i is g. By performing the experiment carefully, we control the variance of X_i to be at most 1. We finally take the experimental value of the acceleration due to gravity to be the average of the values obtained. How many times must we perform the experiment so that we can have 99% confidence that the error is at most 0.01?

Solution: By the weak law of large numbers, if $Y_n = (X_1 + \cdots + X_n)/n$ is the rolling average, then

$$P(|Y_n - g| > 0.01) \leq \frac{1}{n \cdot (0.01)^2}$$

So if we want the latter expression to be less than 1/100 (to have 99% confidence) then we need

$$n \cdot (0.01)^2 \geq 100 \text{ or } n \geq 10^9$$

That is a large number of experiments. That is why we need to control the variance even more. The smaller we can make the variance, the less experiments we need to be confident that we got an accurate result.

Typically, the variance is controlled by the least count of the measuring instruments. Thus, we always seek to imrove those.