Independence, Convergence and LLN

1. We roll a pair of dice, one red and one blue. The random variable X_1 denotes the value on the face of the red die and X_2 denotes the value on the face of the blue die. Which of the following random variables are independent? Justify your answer. In each case compute the covariance and correlation.
   (a) X_1 and X_2.
   (b) $X_1 - X_2$ and $X_1 + X_2$.

2. Given two random variables X and Y with $E(X^2) = 1$ and $E(Y^2) = 0.25$. Which of the following are possible and impossible?
   (a) $E(XY) = 0.25$
   (b) $E(XY) = 0.6$.
   (c) $E(XY) = 0.4$.
   (d) $E(XY) = -0.4$.

3. We repeatedly flip a fair coin. Let X_i denote the random variable that takes the value 1 if the i-th flip returns Head and 0 if it returns Tail. Which of the following statements are True? Justify your answer.
   (a) The sequence X_i converges to 1/2 almost surely.
   (b) The sequence X_i converges to 1/2 in probability.
   (c) The sequence X_i/i converges to 0 in probability.
   (d) The sequence X_i/i converges to 0 almost surely.
   (e) The sequence $Y_n = \sum_{i=1}^{n} X_i$ converges to 1/2 in probability.
   (f) The sequence $S_n = (\sum_{i=1}^{n} X_i)/n$ converges to 1/2 in probability.
   (g) The sequence $S_n = (\sum_{i=1}^{n} X_i)/n$ converges to 1/2 almost surely.
   (h) The sequence $U_n = \sum_{i=1}^{n} X_i/2^i$ converges to 1 in probability.
   (i) The sequence $U_n = \sum_{i=1}^{n} X_i/2^i$ converges to $U = \sum_{i=1}^{\infty} X_i/2^i$ in probability.
   (j) The sequence $U_n = \sum_{i=1}^{n} X_i/2^i$ converges to $U = \sum_{i=1}^{\infty} X_i/2^i$ almost surely.

4. We repeatedly carry out an experiment to measure the acceleration due to gravity. Let X_i be the random variable that denotes the result of the i-th measurement. The expected value of X_i is g. By performing the experiment carefully, we control the variance of X_i to be at most 1. We finally take the experimental value of the acceleration due to gravity to be the average of the values obtained. How many times must we perform the experiment so that we can have 99% confidence that the error is at most 0.01?