Solutions to Quiz 5

1. We are looking for students with birthday on Republic Day. Suppose that there are 365 days in the year and 365 students.

(1 mark) (a) Write the formula for the probability that *at most* 1 student has a birthday on republic day.

(2 marks) (b) Write an approximation (in terms of Euler's constant e) for the probability that *at least* two students have birthday on Republic Day.

(2 marks) (c) Convert the above approximation into a fraction.

Solution: The probability that a randomly chosen student has birthday on Republic Day is given by $p = 1/365$.

Let X be the random variable that counts the number of students out of 365 that have a birtday on Republic Day.

$$P(X = r) = \binom{365}{r} p^r (1-p)^{365-r}$$

Thus, the probablity of at most 1 student is

$$P(X \leq 1) = (1-1/365)^{365} + 365 \cdot (1/365)(1-1/365)^{364} = (1-1/365)^{365} + (1-1/365)^{364}$$

The probability of at least two students is

$$P(X \geq 2) = 1 - P(X \leq 1)$$

Now since 365 is large we can approximate

$$(1-1/365)^{365} + (1-1/365)^{364} = (1-1/365)^{365} \left(1 + (1-1/365)^{-1}\right) \simeq e^{-1} \cdot (1+1) = 2e^{-1}$$

Hence, $P(X \geq 2) = 1 - 2e^{-1}$.

Finally, we can use the series and approximate (as it is an alternating series)

$$1 - 2e^{-1} = 1 - 2(1 - 1 + 1/2 - 1/6 + 1/24 - 1/120 + \dots) \simeq 1 - 2\frac{60 - 20 + 5 - 1}{120} = \frac{4}{15}$$

The cruder approximation

$$1 - 2e^{-1} = 1 - 2(1 - 1 + 1/2 - 1/6 + \dots) \simeq 1 - 2\frac{3-1}{6} = \frac{1}{3}$$

is not too bad either!