Solutions to Assignment 6

1. Use the fact that the series for e^{-1} is an alternating series to estimate it upto second place of decimal. Use this to estimate the probability of at least three successes in a Poisson distribution with $c = 1$. Compare your values to the actual values.

Solution: Since the series $\sum_k (-1)^k/k!$ is alternating, we only need to check for which k the term is $< 1/100$ in absolute value. This happens for $k = 5$. So we need to calculate 6 terms.

$$1 - 1 + 1/2 - 1/6 + 1/24 - 1/120 = 11/30$$

This gives a decimal value of .367 whereas the actual value is about .368.

By the Possion distribution we need to calculate $e^{-1} + e^{-1} + e^{-1}/2$ which is approximately $(5/2) * (11/30) = 11/12$. This turns out to be .917 in decimal compared with the actual value which is about .920.

2. Find the point where the graph of $e^{-x^2/2}$ starts curving upward (putting using second derivative as 0). Estimate the area under this curve upto that point using a "house" shape approximation. Use this to estimate the *odds* that the point lies more than "one-sigma from the mean". (By "odds" one means the ratio of probability of success to the probability of failure.) Compare your values to the actual values.

Solution: We calculate the derivative of $e^{-x^2/2}$ as $-xe^{-x^2/2}$. Hence, the second derivative is $(x^2 - 1)e^{-x^2/2}$. This vanishes for $x = \pm 1$. The shape of $e^{-x^2/2}$ is "like a house" between these two points. From this we can estimate the area under it as area of rectangle of base 2 and height $e^{-1/2}$ plus area of triangle of base 2 and height $1 - e^{-1/2}$; in other words, this is $2e^{-1/2} + (1 - e^{-1/2}) = 1 + e^{-1/2}$. By the alternating series calculations this is about $1 + 29/48$ or $a = 1.60$.

We can make a "better" two-storey house by using a trapezoid of base 2, top 1 and height $e^{-1/8} - e^{-1/2}$ and a triangle of base 1 and height $1 - e^{-1/8}$ to get

$$2e^{-1/2} + (3/2)(e^{-1/8} - e^{-1/2}) + (1/2)(1 - e^{-1/8}) = (1/2)(e^{-1/2} + 2e^{-1/8} + 1)$$

Combined with the alternating series approximation $113/128$ for $e^{-1/8}$, this gives the area estimate as $647/384$ or 1.68. The actual value is 1.71.

The probability that a point lies within one-sigma is $p = a/\sqrt{2\pi}$. Hence, the odds that it lies outside one-sigma is $(1 - p)/p = 1 - 1/p$. We can use the usual 22/7 or 355/113 approximation for π to make the estimates of $\sqrt{2\pi}$.

3. My mother calls me $4(=4k)$ times a week on average and my father calls me $2(=2k)$ times a week on average. If I compare the expected amount of time that I have to wait for $2(=2r)$ calls from my mother to the amount of time I have to wait for one call$(=r)$ from my father, which is greater? Suppose I compare the probability of waiting for at least one week in each case, which is greater? Compare these for other values of k and r.

Solution: The weekly frequency is 4 for calls from my mother. We divide a week into n parts where n is large. The probability of a call in $1/n$ of a week is $4/n$. The probability that I have to wait for exactly $t = r/n$ units (=weeks) for 2 calls is given by the negative binomial distribution

$$\binom{2+r-1}{r}(4/n)^2(1-4/n)^r \text{ with } r = nt$$

The expected value of the negative binomial distribution $NB(k;p)$ is $kp/(1-p)$ which is $2(1-4/n)/(4/n)$. This is the answer in terms of units of $1/n$ weeks so that in terms of weeks this is $2(1-4/n)/4 = (1/2)(1-4/n)$. As n approaches infinity (in other words, we check continuously!), this approaches $1/2$ a week. We similarly calculate that for the father this is also $1/2$ a week.

The probability of waiting for one week or less for my mother to call twice is given by the sum for $r \leq n$.

$$\sum_{r=0}^{n} r(4/n)^2(1-4/n)^r$$

This sum looks a bit difficult to calculate. However, putting $t_r = r/n$, we see that this becomes

$$\sum_{r=0}^{n} t_r 4^2(1-4/n)^{nt_r}(1/n)$$

This is an approximation to the integral

$$\int_0^1 4^2 t \exp(-4t)dt$$

which we calculate to be $1-(e^{-4}+4e^{-4})$. The probability of waiting at least one week becomes $5e^{-4}$. Similarly, hand the probability for waiting at least 1 week for my father to call once is approximated by

$$\int_1^\infty 2\exp(-2t)dt = e^{-2}$$

We easily check that $e^(-2) > 5e^(-4)$.

4. One person does an experiment to measure the mass of an electron and gets the values (in suitable units) 10001, 10001, 10000, 9999, 9999. Another person does the experiment and gets every value $10000+k$ for k in range $[-2,2]$ exactly once. For which experimenter is the confidence greater that the result is 10000 ± 0.5.

Solution: We check that the variance in the second case is bigger than the variance in the first case. Hence, the confidence in the second case for a small interval is less.

5. Estimate (or compute) the probability of getting exactly r Heads in k coin flips for the following values of (k,r). Justify your answers.

   (a) $(10,5)$, $(20,5)$, $(30,5)$, $(40,5)$. What is the trend as k goes to infinity?

   Solution: We see that as k goes to infinity we have

   $$\binom{k}{5}(1/2)^k \simeq \frac{k^5}{2^k 5!}$$

   This goes to 0 exponentially as k goes to infinity.

   (b) $(10,3)$, $(20,6)$, $(30,9)$, $(40,12)$. What is the trend as k goes to infinity?

   Solution: We see that as k goes to infinity we have

   $$\binom{10k}{3k}(1/2)^{10k} \simeq C\sqrt{k}\exp(-x^2/2) \text{ where } x = (5k-3k)/\sqrt{4k}$$

   This clearly goes to 0 like $\exp -ck^2$ for some constant c.

   (c) $(10,2)$, $(40,4)$, $(90,6)$, $(160,8)$. What is the trend as k goes to infinity?

   Solution: We see that as k goes to infinity we have

   $$\binom{10k^2}{2k}(1/2)^{10k^2} \simeq C\sqrt{k}\exp(-x^2/2) \text{ where } x = (5k^2-2k)/k$$

   This too goes to 0 line $\exp(-ck^2)$ for some constant c.

   (d) $(10,2)$, $(20,12)$, $(30,22)$, $(40,32)$. What is the trend as k goes to infinity?

   Solution: We see that

   $$\binom{10k}{10k-8}(1/2)^{10k^2}$$

   This is similar to the first case with 5 replaced by 8.

(e) Does your answer in the above questions change if probability of Head is 0.4 instead of 1/2?

Solution: The constants will change but the trends will not.