The Normal Distribution

If X_k is the random variable that counts the number of Heads in a sequence of k independent coin flips of a coin that returns Head with a probability of p, then X_k follows the Binomial distribution. In other words:

$$P(X_k = r) = \binom{k}{r} p^r (1-p)^{k-r}$$

We have seen earlier that $E(X_k) = kp$ and $\sigma^2(X_k) = kp(1-p)$.

Now, kp goes to infinity as k goes to infinity. Hence, this distribution does not have a nice limit as it stands. Even if we take $Y_k = X_k - kp$, then this variable does not have a nice limit since $\sigma^2(Y_k) = \sigma^2(X_k) = kp(1-p)$ which goes to infinity as k goes to infinity. Hence, we consider will consider the limit of the "normalised" distributions $Z_k = Y_k/\sqrt{kp(1-p)}$ which have the property $E(Z_k) = 0$ and $\sigma^2(Z_k) = 1$ for all k.

Limits of Binomial Distribution

We will make use of the Stirling approximation which says that there is a *constant* C so that:

$$k! \simeq Ck^k e^{-k}\sqrt{k} \text{ as } k \to \infty$$

(Recal the $f(k) \simeq g(k)$ as k goes to infinity means that the ratio of these two functions goes to 1 as k goes to infinity.)

We wish to consider the limit of $P(Z_k = r) = \binom{k}{r} p^r (1-p)^{k-r}$ as k approaches infinity for all those r such that $x = (r - kp)/\sqrt{kp(1-p)}$ remains bounded by some constant A. For each k large enough, we pick such an r and denote it by r_k, we denote the corresponding x as x_k. In order to simplify notation, we use $q = (1-p)$ and we drop the subscripts on r_k and x_k (but we should not forget the dependence!). Our claim is that

$$\binom{k}{r} p^r q^{k-r} \simeq \frac{e^{-x^2/2}}{C\sqrt{kpq}}$$

as k approaches infinity. Note that the right hand side is independent of the chosen constant A, but we need to fix A in order to obtain this asymptotic formula.

First of all we note that $r = kp + x\sqrt{kpq}$ and easily calculate that $k - r = kq - x\sqrt{kpq}$. Since $\sqrt{k} = o(k)$ for k going to infinity and $|x| < A$ remains bounded, we see that $r \simeq kp$ and $k - r \simeq kq$ as k goes to infinity. In particular, r and $k - r$ must both go to infinity as well. Hence, we can use the Stirling approximation for k, r and $k - r$. This gives

$$\binom{k}{r} p^r q^{k-r} \simeq (1/C) \frac{k^k e^{-k}\sqrt{k} p^r q^{k-r}}{(r^r e^{-r}\sqrt{r}) \cdot \left((k-r)^{k-r} e^{-(k-r)}\sqrt{k-r}\right)}$$

Now the numerator can be "separated" using the following:

$$k^k p^r q^{k-r} = (kp)^r (kq)^{k-r} \text{ and } e^{-k} = e^{-r} \cdot e^{-(k-r)}$$

We use these to get

$$\binom{k}{r} p^r q^{k-r} \simeq (1/C)\sqrt{\frac{k}{r(k-r)}} \left(\frac{kp}{r}\right)^r \left(\frac{kq}{k-r}\right)^{k-r}$$

Now we substitute

$$kp = r - x\sqrt{kpq} \text{ and } kq = (k-r) + x\sqrt{kpq}$$

in the latter two terms to get

$$\left(\frac{kp}{r}\right)^r = \left(1 - \frac{x\sqrt{kpq}}{r}\right)^r \text{ and } \left(\frac{kq}{k-r}\right)^{k-r} = \left(1 + \frac{x\sqrt{kpq}}{k-r}\right)^{k-r}$$

Now $x\sqrt{kpq}/r \simeq x\sqrt{q/kp} \to 0$ as k goes to infinity. Similarly $x\sqrt{kpq}/(k-r) \to 0$ as k goes to infinity. Hence, for large enough k we can use the approximation $\log(1+t) = t - t^2/2 + g(t)$ with $|g(t)| < t^3/3$ to get

$$\log\left(\frac{kp}{r}\right)^r = r\left(-\frac{x\sqrt{kpq}}{r} - \frac{x^2 kpq}{2r^2} + g(-x\sqrt{kpq}/r)\right)$$

and

$$\log\left(\frac{kq}{k-r}\right)^{k-r} = (k-r)\left(\frac{x\sqrt{kpq}}{k-r} - \frac{x^2 kpq}{2(k-r)^2} + g(x\sqrt{kpq}/(k-r))\right)$$

The first terms cancel, while the second terms give

$$-\frac{x^2 k^2 pq}{2r(k-r)}$$

Finally, we have

$$|rg(-x\sqrt{kpq}/r) + (k-r)g(x\sqrt{kpq}/(k-r))| \leq (A^3/3)\left(\frac{(kpq)^{3/2}}{r^2} + \frac{(kpq)^{3/2}}{(k-r)^2}\right)$$

Now, kp/r goes to 1 as k goes to infinity so $(kpq)^{3/2}/r^2$ goes to $q^{3/2}/r^{1/2}$ as k goes to infinity. Since r goes to infinity as k goes to infinity, we see that $(kpq)^{3/2}/r^2$ goes to 0 as k goes to infinity. Similarly, $(kpq)^{3/2}/(k-r)^2$ goes to 0 as k goes to infinity.

Combining the above calculations, we get

$$\left(\frac{kp}{r}\right)^r \cdot \left(\frac{kq}{k-r}\right)^{k-r} \simeq \exp\left(-\frac{x^2 k^2 pq}{2r(k-r)}\right)$$

as k goes to infinity. Putting it all together

$$\binom{k}{r} p^r q^{k-r} \simeq (1/C)\sqrt{\frac{k}{r(k-r)}}\exp(-\frac{x^2k^2pq}{2r(k-r)}$$

Finally, we use $r \simeq kp$ and $k - r \simeq kq$ to simplify this to

$$\binom{k}{r} p^r q^{k-r} \simeq (1/C)\sqrt{\frac{1}{kpq}}\exp\left(-\frac{x^2}{2}\right)$$