① $$\frac{1}{H_0^2}\frac{\dot{a}^2}{a^2} = \Omega_{nr}\left(\frac{a_0}{a}\right)^3 + (1-\Omega_{nr})\left(\frac{a_0}{a}\right)^2$$

$$\Omega_{nr} > 1$$

$$y = \frac{a}{a_0} \quad , \quad x = tH_0$$

$$\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = \frac{\Omega_{nr}}{y^3} - (\Omega_{nr}-1)\frac{1}{y^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\Omega_{nr}}{y} - (\Omega_{nr}-1)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\Omega_{nr}}{y}\left[1 - \frac{(\Omega_{nr}-1)}{\Omega_{nr}}y\right]$$

$$\frac{y^{1/2}\,dy}{\Omega_{nr}^{1/2}\left[1 - \frac{(\Omega_{nr}-1)}{\Omega_{nr}}y\right]^{1/2}} = dx$$

$$\frac{\Omega_{nr}-1}{\Omega_{nr}}\, y \leq 1$$

$$\Rightarrow y_{max} = \frac{\Omega_{nr}}{\Omega_{nr}-1}$$

$$y = y_{max} \sin^2\left(\frac{\theta}{2}\right)$$

$$dy = y_{max} \sin\frac{\theta}{2} \cos\frac{\theta}{2}\, d\theta$$

$$y^{1/2} = y_{max}^{1/2} \sin\frac{\theta}{2}$$

$$\Rightarrow dx = \frac{1}{\Omega_{m}^{1/2}} \cdot \frac{y_{max}^{3/2} \sin^2\frac{\theta}{2} \cos\frac{\theta}{2}\, d\theta}{\left[1 - \sin^2\frac{\theta}{2}\right]^{1/2}}$$

$$= \frac{y_{max}^{3/2}}{\Omega_{m}^{1/2}} \cdot \sin^2\frac{\theta}{2}\, d\theta$$

$$= \frac{1}{2} \frac{y_{max}^{3/2}}{\Omega_{m}^{1/2}} (1 - \cos\theta)\, d\theta$$

$$\Rightarrow tH_0 + \text{Const} = \frac{1}{2} \frac{y_{max}^{3/2}}{\Omega_{m}^{1/2}} (\theta - \sin\theta)$$

$$y = \frac{y_{max}}{2} (1 - \cos\theta)$$

$$a = 0 \Rightarrow y = 0 \Rightarrow \theta = 0$$

if $t = 0$ @ $a = 0$, $\text{Const} = 0$

$$\Rightarrow \boxed{\begin{aligned} \frac{a}{a_0} &= \frac{1}{2}\frac{\Omega_{m}}{(\Omega_{m}-1)}(1-\cos\theta) \\ t &= \frac{1}{2H_0}\frac{\Omega_{m}}{(\Omega_{m}-1)^{3/2}}(\theta-\sin\theta) \end{aligned}}$$

early times $\Rightarrow$ $t \ll t_0 \Rightarrow \theta \ll \pi$

taking $\theta \ll 1$

$$\lim_{\theta \ll 1} \frac{a}{a_0} = \frac{1}{2}\frac{\Omega_{m}}{\Omega_{m}-1}\left(1-\left(1-\frac{\theta^2}{2}\right)\right)$$

$$= \frac{1}{4}\frac{\Omega_{m}}{\Omega_{m}-1}\theta^2$$

$$\lim_{\theta \ll 1} t = \frac{1}{2H_0}\frac{\Omega_{m}}{(\Omega_{m}-1)^{3/2}}\left[\theta-\left(\theta-\frac{\theta^3}{6}\right)\right]$$

$$= \frac{1}{12H_0}\frac{\Omega_{m}}{(\Omega_{m}-1)^{3/2}}\theta^3 \propto \left(\frac{a}{a_0}\right)^{3/2}$$

$\Rightarrow a \propto t^{2/3}$ at early times.

2) $\cancel{3}$ $\rho_c = \dfrac{3H_0^2}{8\pi G}$, $H_0 = 70$ km/s/Mpc

$H_0 = 70$ km/s/Mpc

$$= \frac{70 \times 10^3}{3.08 \times 10^{22}} = \frac{7}{3.08} \times 10^{-18}$$

$$= 2.27 \times 10^{-18}\ s^{-1}$$

$$\Rightarrow \rho_c = \frac{3 \times (2.27)^2 \times 10^{-36}}{8 \times 3.14 \times 6.7 \times 10^{-11}}$$

$$= \frac{3 \times (2.27)^2}{8 \times 3.14 \times 6.7} \times 10^{-25}\ \frac{Kg}{m^3}$$

$$= 9.2 \times 10^{-27}\ Kg/m^3$$

(3) $\Omega_H = 0.045$

$$\Rightarrow \rho_H = 0.045 \times 9.2 \times 10^{-27} \ kg/m^3$$

$$= 4.1 \times 10^{-28} \ Kg/m^3$$

$$\Rightarrow n_H = \frac{4.1 \times 10^{-28}}{m_H} = \frac{4.1 \times 10^{-28}}{1.67 \times 10^{-27}}$$

$$= 0.025 \ m^{-3}$$

(4) $$H^2 = H_0^2 \left[\Omega_m (1+z)^3 + (1-\Omega_m)(1+z)^2 \right]$$

$$\rho_c(z) = \frac{3H^2}{8\pi G}$$

$$\rho_m(z) = \rho_m(z=0)\,(1+z)^3$$

$$\Omega_m(z) = \frac{\rho_m(z)}{\rho_c(z)} = 8\pi G \frac{\rho_m (1+z)^3}{3H^2}$$

$$= \frac{H_0^2\,\Omega_m (1+z)^3}{H^2}$$

$$= \frac{\Omega_m (1+z)^3}{\Omega_m (1+z)^3 + (1-\Omega_m)(1+z)^2}$$

(5) $$\Omega_m(z) = \frac{\Omega_m (1+z)^3}{\Omega_m (1+z)^3 + (1-\Omega_m)(1+z)^2}$$

$$1 - \Omega_m(z) = \frac{\cancel{\Omega_m (1+z)^3} + (1-\Omega_m)(1+z)^2 - \cancel{\Omega_m (1+z)^3}}{\Omega_m (1+z)^3 + (1-\Omega_m)(1+z)^2}$$

$$= \frac{1-\Omega_m}{\Omega_m (1+z) + 1 - \Omega_m}$$

$$= \frac{1-\Omega_m}{1+\Omega_m z}$$

as $t \to 0$, $z \to \infty$

$\Rightarrow 1 - \Omega_m(z) \to 0$

(6) $$\Phi(L)\,dL = \Phi_* \left(\frac{L}{L_*}\right)^{-\alpha} e^{-L/L_*} \frac{dL}{L_*}$$

(a)

n : Number density $= \int_0^\infty \Phi(L)\,dL$

$$= \Phi_* \int_0^\infty \left(\frac{L}{L_*}\right)^{-\alpha} e^{-L/L_*} \frac{dL}{L_*}$$

$$y = \frac{L}{L_*}, \quad dy = \frac{dL}{L_*}$$

$$\Rightarrow n = \Phi_* \int_0^\infty y^{-\alpha} e^{-y}\,dy$$

$$\Rightarrow \boxed{n = \Phi_* \Gamma(1-\alpha)}$$

(b) l : Luminosity density

$$l = \int_0^\infty L\,\Phi(L)\,dL = \Phi_* \int_0^\infty L \left(\frac{L}{L_*}\right)^{-\alpha} e^{-L/L_*} \frac{dL}{L_*}$$

$$= \Phi_* L_* \underbrace{\int_0^\infty y^{1-\alpha} e^{-y}\,dy}$$

$$\boxed{l = \Phi_* L_* \Gamma(2-\alpha)}$$