Solutions to Assignment 3

1. Recall the experiment from Lecture 4 which was as follows.

   1. We flip three coins.
   2. If all three coins are different, we record the occurrence of events as follows: 1st event for the sequence (T, H, H), 2nd event for (H, T, H), 3rd event for (H, H, T), 4th event for (H, T, T), 5th event for (T, H, T) and 6th event for (T, T, H).
   3. All three coins are the same, then we go back to step 1 and continue!

   Calculate the probability of getting the result (T, H, H) using the limit law for probability.

> **Solution:** Let S_k be the event that on the k-flip of three coins, all are the same; we have $P(S_k) = 2/8 = 1/4$. Let T_k be the event that on the k-th flip of three coins we get (T, H, H); we have $P(T_k) = 1/8$. The event A_k that results in (T, H, H) being recorded is $S_1 \cap \ldots S_{k-1} \cap T_k$. By the independence of these events, we have $P(A_k) = (1/4)^{k-1} \cdot (1/8)$.
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> The event that (T, H, H) is recorded is the same as $B = \cup_n A_n$. Let $B_n = A_1 \cup \ldots A_n$. Since A_n are mutually exclusive events, $P(B_n) = \sum_{k=1}^{n} (1/4)^{n-1} \cdot (1/8)$. By the limit law
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> $$P(B) = \sup_n P(B_n) = \sum_{k=0}^{\infty} (1/4)^{n-1} \cdot (1/8) = (1/8) \cdot \frac{1}{1 - (1/4)} = 1/6$$
>
> (Note that this is what justifies the replacement of the roll of dice by this coin experiment.)

2. We repeated roll a pair of dice until the sum is *not* 7. Find the probability of each possible sum (from 2 to 12).

> **Solution:** For the standard roll of a pair of dice, the events S_k for the sum being k

correspond to pairs of rolls given below:

$$\begin{aligned}
S_2 &\leftrightarrow \{(1,1)\} \\
S_3 &\leftrightarrow \{(1,2),(2,1)\} \\
S_4 &\leftrightarrow \{(1,3),(2,2),(3,1)\} \\
S_5 &\leftrightarrow \{(1,4),(2,3),(3,2),(4,1)\} \\
S_6 &\leftrightarrow \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \\
S_7 &\leftrightarrow \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \\
S_8 &\leftrightarrow \{(2,6),(3,5),(4,4),(5,3),(6,2)\} \\
S_9 &\leftrightarrow \{(3,6),(4,5),(5,4),(6,3)\} \\
S_{10} &\leftrightarrow \{(4,6),(5,5),(6,4)\} \\
S_{11} &\leftrightarrow \{(5,6),(6,5)\} \\
S_{12} &\leftrightarrow \{(6,6)\}
\end{aligned}$$

It follows that the probabilities are

$$\begin{aligned}
P(S_2) &= 1/36 \\
P(S_3) &= 2/36 \\
P(S_4) &= 3/36 \\
P(S_5) &= 4/36 \\
P(S_6) &= 5/36 \\
P(S_7) &= 6/36 \\
P(S_8) &= 5/36 \\
P(S_9) &= 4/36 \\
P(S_{10}) &= 3/36 \\
P(S_{11}) &= 2/36 \\
P(S_{12}) &= 1/36
\end{aligned}$$

In the given experiment, we keep rolling until we do not get 7. Consider the event $T_{k,n}$ that we get a non-7 for the first time on the n-th roll and the resulting sum is k. Probability of $T_{k,n}$ is the same as the probability of 7 on $n-1$ rolls followed by k on the n-th roll. By independence of the events this is $(1/6)^{n-1} \cdot P(S_k)$. Now all the events $T_{k,n}$ are mutually exclusive. Hence the probability of the event $T_k = \cup_n T_{k,n}$ is given by

$$P(T_k) = \sum_{n=1}^{\infty} (1/6)^{n-1} P(S_k) = P(S_k) \cdot (6/5)$$

It follows that we have

$$\begin{aligned}
P(T_2) &= 1/30 \\
P(T_3) &= 2/30 \\
P(T_4) &= 3/30 \\
P(T_5) &= 4/30 \\
P(T_6) &= 5/30 \\
P(T_8) &= 5/30 \\
P(T_9) &= 4/30 \\
P(T_{10}) &= 3/30 \\
P(T_{11}) &= 2/30 \\
P(T_{12}) &= 1/30
\end{aligned}$$

3. Given a real-valued random variable X, what is the relation between $P(X \leq 0)$ and the probabilities $P(X \leq 1/n)$ for all positive integer values of n. Is there anything special about 0 in this example?

Solution: If A_n denotes the event $X \leq 1/n$ then the intersection of all of these $A = \cap_n A_n$ is the same as $X \leq 0$. Since the A_n are decreasing (non-increasing) it follows that

$$P(X \leq 0) = \inf_n P(X \leq 1/n)$$

We can replace 0 by any real number c, and $1/n$ by $c+1/n$ the same reasoning works to give

$$P(X \leq c) = \inf_n P(X \leq c + 1/n)$$

4. Given a real-valued random variable X what is the relation between $P(X \leq 0$ and the probabilities $P(X \leq -1/n)$ for all positive integer values of n. Is there anything special about 0 in this example?

Solution: If A_n denotes the event $X \leq -1/n$ then the union of all of these $A = \cap_n A_n$ is the same as $X \leq 0$. Since the A_n are increasing (non-decreasing) it follows that

$$P(X < 0) = \sup_n P(X \leq -1/n)$$

Moreover, we have $X \leq 0$ as the disjoint union of the events $X < 0$ and $X = 0$, hence we have

$$P(X \leq 0) = P(X = 0) + \sup_n P(X \leq -1/n)$$

We can replace 0 by any real number c, and $1/n$ by $c+1/n$ the same reasoning works to give

$$P(X \leq c) = P(X = c) + \sup_n P(X < c - 1/n)$$

5. In a repeatable experiment Σ there are events A and B which can be observed. Assume that $P(B) > 0$. We now carry out Σ repeatedly until B is observed. What is the probability that A is also observed at the same time as B?

Solution: Let T_n denote the event that $A \cap B$ is observed on the n-th repetition of Σ while B^c is observed on all $n-1$ earlier repetitions of Σ. The probability of T_n is $P(B^c)^{n-1}P(A \cap B) = (1 - P(B))^{n-1}P(A \cap B)$. Let T be the event that $A \cap B$ is observed after repeatedly observing B^c; this is the union of T_n over all n. Since the events T_n are mutually exclusive, we see that

$$P(T) = \sum_{n=1}^{\infty}(1 - P(B))^{n-1}P(A \cap B) = P(A \cap B)/P(B) = P(A|B)$$

Thus, this repeated experimentation until we see B is an interpretation of conditional probability of A given B.

6. We know that 40% of the fish in a pond are female. We repeatedly catch and release fish until we find one that is female; in that case we record its species before releasing it. After 100 such recordings we find that 20 of these females were pomfret fish. What is a reasonable estimate of the percentage of female pomfret in the pond? Can we conclude the 20% of the fish are pomfret?

Solution: Let $P(FP)$ denote the probability of a randomly chosen fish being a female pomfret. The probability that we pick such a fish on the n-th trial is the same as the probability that we pick $n-1$ male fish followed by this fish; in other words it is $(1-0.4)^{n-1}P(FP)$. It follows that the probability of recording a pomfret by the mechanism used is $\sum_n (0.6)^{n-1}P(F \cap P) = (1/0.4)P(FP)$. By the above frequency, of 20/100 we can estimate $0.2 = (1/0.4)P(FP)$ or $P(FP) = 0.08$. In other words, we estimate that 8% of the fish are female pomfret.

We *cannot* conclude that 20% of the fish are pomfret. We must *also* do a similar experiment where we only record the species of male fish, suppose that in this case

30% of the fish turn out to be pomfret. From a calculation like the one above we will get the probability of male pomfret as 0.18. Putting these together we would get the percentage of pomfret as 26%.