Friedman's equations from Robertson Walker metric

January 29, 2016

1 Metric components

Various components of Robertson-Walker metric are given below.

$$g_{00} = 1$$

$$g_{11} = -\frac{a^2(t)}{1 - kr^2}$$

$$g_{22} = -a^2(t)r^2$$

$$g_{33} = -a^2(t)r^2\sin^2\theta$$

Or their contravariant components

$$g^{00} = 1$$

$$g^{11} = -\frac{(1 - kr^2)}{a^2(t)}$$

$$g^{22} = -\frac{1}{a^2(t)r^2}$$

$$g^{33} = -\frac{1}{a^2(t)r^2\sin^2\theta}$$

derivatives of metric components are given below.

$$g_{00,0} = g_{00,1} = g_{00,2} = g_{00,3} = 0$$

$$g_{11,0} = -\frac{2a\dot{a}}{(1 - kr^2)} = 2\frac{\dot{a}}{a}g_{11}$$

$$g_{11,1} = \frac{-2kra^2}{(1 - kr^2)^2} = \frac{2kr}{(1 - kr^2)}g_{11}$$

$$g_{11,2} = g_{11,3} = 0$$

$$g_{22,0} = 2\frac{\dot{a}}{a}g_{22} = -2a\dot{a}r^2$$

$$g_{22,1} = \frac{2}{r} g_{22} = -2a^2 r$$

$$g_{22,} = g_{22,3} = 0$$

$$g_{33,0} = \frac{\dot{a}}{a} g_{33} = -2a\dot{a}r^2 \sin^2\theta$$

$$g_{33,1} = \frac{2}{r} g_{33} = -2a^2 r \sin^2\theta$$

$$g_{33,2} = 2\cot\theta g_{33} = -2a^2 r^2 \sin\theta \cos\theta$$

$$g_{33,3} = 0$$

2 Chrtistofel's coficents

We know Christofel's connections are given by

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda}(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\gamma\beta,\lambda})$$

various components of Christofel's connections are given below

$$\Gamma^0_{00} = \Gamma^0_{10} = \Gamma^0_{20} = \Gamma^0_{30} = \Gamma^0_{12} = \Gamma^0_{13} = \Gamma^0_{23} = 0$$

$$\Gamma^0_{11} = \frac{\dot{a}}{a} g_{11} = \frac{a\dot{a}}{(1 - kr^2)}$$

$$\Gamma^0_{22} = -\frac{\dot{a}}{a} g_{22} = a\dot{a}r^2$$

$$\Gamma^0_{33} = \frac{\dot{a}}{a} g_{33} = a\dot{a}r^2 \sin^2\theta$$

$$\Gamma^1_{00} = \Gamma^1_{02} = \Gamma^1_{03} = \Gamma^1_{21} = \Gamma^1_{31} = \Gamma^1_{23} = 0$$

$$\Gamma^1_{01} = \frac{\dot{a}}{a}$$

$$\Gamma^1_{22} = -r(1 - kr^2)$$

$$\Gamma^1_{11} = \frac{kr}{(1 - kr^2)}$$

$$\Gamma^1_{33} = -r(1 - kr^2)\sin^2\theta$$

$$\Gamma^2_{00} = \Gamma^2_{01} = \Gamma^2_{03} = \Gamma^2_{11} = \Gamma^2_{22} = \Gamma^2_{13} = \Gamma^2_{23} = 0$$

$$\Gamma^2_{02} = \frac{\dot{a}}{a}$$

$$\Gamma^2_{33} = -\sin\theta \cos\theta$$

$$\Gamma^2_{12} = \frac{1}{r}$$

$$\Gamma^3_{00} = \Gamma^3_{01} = \Gamma^3_{02} = \Gamma^3_{21} = \Gamma^3_{22} = \Gamma^3_{33} = \Gamma^3_{1} = 0$$

$$\Gamma^3_{03} = \frac{\dot{a}}{a}$$

$$\Gamma^3_{23} = -\cot\theta$$

$$\Gamma^3_{13} = \frac{1}{r}$$

3 Ricci tensor

We know that

$$R_{\alpha\beta} = \Gamma^{\sigma}_{\alpha\beta,\sigma} - \Gamma^{\sigma}_{\alpha\sigma,\beta} + \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{\alpha\rho}\Gamma^{\rho}_{\beta\sigma}$$

So

$$R_{00} = \Gamma^{\sigma}_{00,\sigma} - \Gamma^{\sigma}_{0\sigma,0} + \Gamma^{\sigma}_{00}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{0\rho}\Gamma^{\rho}_{0\sigma}$$
$$= -\Gamma^{1}_{01,0} - \Gamma^{2}_{02,0} - \Gamma^{3}_{03,0} - \Gamma^{1}_{01}\Gamma^{1}_{01} - \Gamma^{2}_{02}\Gamma^{2}_{02} - \Gamma^{3}_{03}\Gamma^{3}_{03}$$
$$= -3(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}) - 3\frac{\dot{a}^2}{a^2} = -3\frac{\ddot{a}}{a}$$
$$R_{11} = \Gamma^{\sigma}_{11,\sigma} - \Gamma^{\sigma}_{1\sigma,1} + \Gamma^{\sigma}_{11}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{1\rho}\Gamma^{\rho}_{1\sigma}$$
$$= \Gamma^{0}_{11,0} - \Gamma^{2}_{12,1} - \Gamma^{3}_{13,1} + \Gamma^{0}_{11}(\Gamma^{1}_{01} + \Gamma^{2}_{02} + \Gamma^{3}_{03}) +$$
$$\Gamma^{1}_{11}(\Gamma^{1}_{11} + \Gamma^{2}_{12} + \Gamma^{3}_{13}) - 2\Gamma^{o}_{11}\Gamma^{1}_{10} - \Gamma^{1}_{11}\Gamma^{1}_{11} - \Gamma^{2}_{12}\Gamma^{2}_{12} - \Gamma^{3}_{13}\Gamma^{3}_{13}$$
$$= \partial_0[\frac{a\dot{a}}{(1-kr^2)}] - 2\partial_1[\frac{1}{r}] + \frac{a\dot{a}}{(1-kr^2)}[\frac{3\dot{a}}{a}] +$$
$$\frac{kr}{(1-kr^2)}[\frac{kr}{(1-kr^2)} + \frac{2}{r}] - \frac{2a\dot{a}}{(1-kr^2)}[\frac{\dot{a}}{a}] - [\frac{kr}{(1-kr^2)}]^2 - 2\frac{1}{r^2}$$
$$= \frac{a\ddot{a}}{(1-kr^2)} + \frac{\dot{a}^2}{(1-kr^2)} + 3\frac{\dot{a}^2}{(1-kr^2)} + \frac{2k}{(1-kr^2)} - 2\frac{\dot{a}^2}{(1-kr^2)}$$
$$= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{(1-kr^2)}$$

next component of Ricci tensor is

$$R_{22} = \Gamma^{\sigma}_{22,\sigma} - \Gamma^{\sigma}_{2\sigma,2} + \Gamma^{\sigma}_{22}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{2\rho}\Gamma^{\rho}_{2\sigma}$$
$$= \Gamma^{0}_{22,0} + \Gamma^{1}_{22,1} - \Gamma^{3}_{23,2} + \Gamma^{0}_{22}(\Gamma^{1}_{01} + \Gamma^{2}_{02} + \Gamma^{3}_{03}) + \Gamma^{1}_{22}(\Gamma^{1}_{11} + \Gamma^{2}_{12} + \Gamma^{3}_{13})$$
$$-2\Gamma^{0}_{22}\Gamma^{2}_{02} - 2\Gamma^{1}_{22}\Gamma^{2}_{21} - \Gamma^{3}_{23}\Gamma^{3}_{23}$$
$$= a\ddot{a}r^2 + \dot{a}^2r^2 - 1 + 3kr^2 + cosec^2\theta + a\dot{a}r^2[\frac{3\dot{a}}{a}] - r(1-kr^2)[\frac{kr}{(1-kr^2)} + \frac{2}{r}]$$
$$-2a\dot{a}r^2\frac{\dot{a}}{a} + 2(1-kr^2) - \cot^2\theta$$
$$= a\ddot{a}r^2 + 2a\dot{a}r^2 + 2kr^2$$
$$= r^2(a\ddot{a} + 2a^2\dot{a}^2 + 2k)$$

next component

$$R_{33} = \Gamma^{\sigma}_{33,\sigma} - \Gamma^{\sigma}_{3\sigma,3} + \Gamma^{\sigma}_{33}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\sigma}_{3\rho}\Gamma^{\rho}_{3\sigma}$$
$$= \Gamma^{0}_{33,0} + \Gamma^{1}_{33,1} + \Gamma^{2}_{33,2} + \Gamma^{0}_{33}(\Gamma^{1}_{01} + \Gamma^{2}_{02} + \Gamma^{3}_{03}) +$$
$$\Gamma^{1}_{33}(\Gamma^{1}_{11} + \Gamma^{2}_{12} + \Gamma^{3}_{13}) - 2\Gamma^{0}_{33}\Gamma^{3}_{30} - 2\Gamma^{1}_{33}\Gamma^{3}_{31} - \Gamma^{3}_{32}2\Gamma^{2}_{33}$$
$$= a\ddot{a}r^2\sin^2\theta + \dot{a}^2r^2\sin^2\theta + 3kr^2\sin^2\theta - \sin^2\theta + \sin^2\theta -$$
$$\cos^2\theta + \dot{a}ar^2\sin^2\theta[\frac{3\dot{a}}{a}] + (-r(1-kr^2)\sin^2\theta)[\frac{kr}{(1-kr^2)} + \frac{2}{r}]$$

$$-2a\dot{a}r^2\sin^2\theta[\frac{\dot{a}}{a}] - 2(-r(1-kr^2)\sin^2\theta)[\frac{1}{-}] - (-\cos\theta\sin\theta)[\cot\theta]$$
$$= r^2\sin^2\theta(a\ddot{a} + 2a^2\dot{a}^2 + 2k)$$

now we know

$$R_{00} = -3\frac{\ddot{a}}{a}$$
$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{(1-kr^2)}$$
$$R_{22} = r^2(a\ddot{a} + 2a^2\dot{a}^2 + 2k)$$
$$R_{33} = r^2\sin^2\theta(a\ddot{a} + 2a^2\dot{a}^2 + 2k)$$

4 Ricci scalar

Ricci scalar is defined as following

$$R = R_{11}g^{11} + R_{22}g^{22} + R_{33}g^{33}$$

putting the values of metric cofficents from section (1) and values of components of Ricci tensor from section (3) .

$$R = (\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{(1-kr^2)})(-\frac{(1-kr^2)}{a^2(t)}) +$$
$$(r^2(a\ddot{a} + 2a^2\dot{a}^2 + 2k))(-\frac{1}{a^2(t)r^2}) +$$
$$(r^2\sin^2\theta(a\ddot{a} + 2a^2\dot{a}^2 + 2k))(-\frac{1}{a^2(t)r^2\sin^2\theta})$$
$$= -\frac{6}{a^2}[a\ddot{a} + \dot{a}^2 + k]$$

5 Friedman's equations

We know the Einstein's equations are

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}(R + \Lambda) = 8\pi G T_{\alpha\beta}$$

zeroth component of this equation is

$$R_{00} - \frac{1}{2}g_{00}(R + \Lambda) = 8\pi G T_{00}$$

we know that

$$T_{\alpha\beta} = (p + \rho)U_\alpha U_\beta - p g_{\alpha\beta}$$

so

$$T_{00} = \rho$$

and

$$T_{11} = -g_{11}p = \frac{a^2}{(1-kr^2)}p$$

now zeroth component of the Einstein's equation can be written as

$$-3\frac{\ddot{a}}{a} - \frac{1}{2}(-\frac{6}{a^2}[a\ddot{a} + \dot{a}^2 + k] + \Lambda) = 8\pi G\rho$$

or

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{6} \tag{1}$$

th Einstein's equation is

$$R_{11} - \frac{1}{2}g_{11}(R + \Lambda) = 8\pi G T_{11}$$

or

$$\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{(1-kr^2)} - \frac{1}{2}(-\frac{a^2(t)}{1-kr^2})(-\frac{6}{a^2}[a\ddot{a} + \dot{a}^2 + k] + \Lambda)$$

$$= 8\pi G(\frac{a^2}{(1-kr^2)}p)$$

or

$$a\ddot{a} + 2\dot{a}^2 + 2k - 3(a\ddot{a} + \dot{a}^2 + k) + \Lambda\frac{a^2}{2} = 8\pi G a^2 p$$

or

$$-2a\ddot{a} - (\dot{a}^2 + k) = +8\pi G a^2 p - \Lambda\frac{a^2}{2}$$

using equation (1) we get

$$-2a\ddot{a} - a^2(\frac{8\pi G\rho}{3} + \frac{\Lambda}{6}) = 8\pi G a^2 p - \Lambda\frac{a^2}{2}$$

or

$$-2a\ddot{a} = 8\pi G a^2[p + \frac{\rho}{3}] - \frac{\Lambda}{3}a^2$$

or

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}(\rho + 3p) + \frac{\Lambda}{6} \tag{2}$$

equation (1) and (2) are called Friedman's equation.